

Weak turbulence theory for rotating magnetohydrodynamics and planetary dynamos

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A weak turbulence theory is derived for magnetohydrodynamics under rapid rotation and in the presence of a large-scale magnetic field. The angular velocity Ω_0 is assumed to be uniform and parallel to the constant Alfvén speed \mathbf{b}_0 . Such a system exhibits left and right circularly polarized waves which can be obtained by introducing the magneto-inertial length $d \equiv b_0/\Omega_0$. In the large-scale limit ($kd \rightarrow 0$; k being the wave number), the left- and right-handed waves tend respectively to the inertial and magnetostrophic waves whereas in the small-scale limit ($kd \rightarrow +\infty$) pure Alfvén waves are recovered. By using a complex helicity decomposition, the asymptotic weak turbulence equations are derived which describe the long-time behavior of weakly dispersive interacting waves *via* three-wave interaction processes. It is shown that the nonlinear dynamics is mainly anisotropic with a stronger transfer perpendicular (\perp) than parallel (\parallel) to the rotating axis. The general theory may converge to pure weak inertial/magnetostrophic or Alfvén wave turbulence when the large or small-scales limits are taken respectively. Inertial wave turbulence is asymptotically dominated by the kinetic energy/helicity whereas the magnetostrophic wave turbulence is dominated by the magnetic energy/helicity. For both regimes a family of exact solutions are found for the spectra which do not correspond necessarily to a maximal helicity state. It is shown that the hybrid helicity exhibits a cascade whose direction may vary according to the scale k_f at which the helicity flux is injected with an inverse cascade if $k_f d < 1$ and a direct cascade otherwise. The theory is relevant for the magnetostrophic dynamo whose main applications are the Earth and giant planets for which a small ($\sim 10^{-6}$) Rossby number is expected.

Key words: Dynamo theory, MHD turbulence, wave-turbulence interactions

1. Introduction

Rotation is a commonly observed phenomenon in astronomy: planets, stars and galaxies all spin around their axis. The rotation rate of planets in the solar system was first measured by tracking visual features whereas stellar rotation is generally measured through Doppler shift or by following the magnetic activity. One consequence of the Sun rotation is the formation of the Parker interplanetary magnetic field spiral well detected by space crafts, whereas the Earth rotation has a strong impact on the turbulent dynamics of large-scale geophysical flows. These few examples show that the study of rotating flows interests a wide range of problems, ranging from engineering (turbomachinery) to geophysics (oceans, earth's atmosphere, gaseous planets), weather prediction and turbulence

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(Davidson 2004). Rotation is often coupled with other dynamical factors, it is therefore important to isolate the effect of the Coriolis force to understand precisely its impact. The importance of rotation can be measured with the Rossby number:

$$Ro = \frac{U_0}{L_0 \Omega_0}, \quad (1.1)$$

where U_0 , L_0 and Ω_0 are respectively typical velocity, length-scale and rotation rate. This dimensionless number measures the ratio of the advection term on the Coriolis force in the Navier-Stokes equations, also a small value of Ro means a dynamics mainly driven by rotation. Typical large-scale planetary flows are characterized by $Ro \sim 0.1$ (Shirley & Fairbridge 1997) whereas the liquid metals (mainly iron) in the Earth's outer core are much more affected by rotation with $Ro \sim 10^{-6}$ (Roberts & King 2013). Note that for a giant planet like Jupiter in which liquid metallic hydrogen is present in most of the volume, it is believed that the Rossby number may even be smaller (see e.g. Jones 2011). These situations contrast with the solar convective region where the magnetic field is believed to be magnified and for which $Ro \sim 1$.

Inertial waves are a ubiquitous feature of neutral fluids under rapid rotation (Greenspan 1968). Although much is known about their initial excitation, their nonlinear interactions is still a subject of intense research. Many papers have been devoted to pure rotating turbulence ($Ro \leq 1$) but because of the different nature of the investigations (theoretical, numerical and experimental) it is difficult in general to compare directly the results obtained. From a theoretical point of view it is convenient to use a spectral description in terms of continuous wave vectors with the unbounded homogeneity assumption in order to derive the governing equations for the energy, kinetic helicity and polarization spectra (Cambon & Jacquin 1989). Although such equations introduce transfer terms which remain to be evaluated consistently, it is already possible to show with a weakly nonlinear resonant waves analysis (Waleffe 1993) the anisotropic nature of that turbulence with a nonlinear transfer preferentially in the perpendicular (to $\mathbf{\Omega}_0 = \Omega_0 \mathbf{e}_\parallel$) direction. For moderate Rossby numbers the eddy damped quasi-normal Markovian model may be used as a closure (Cambon *et al.* 1997), whereas in the small Rossby number limit the asymptotic weak turbulence theory can be derived rigorously (Galtier 2003). In the latter case, it was shown that the wave modes ($k_\parallel > 0$) are decoupled from the slow mode ($k_\parallel = 0$) which is not accessible by the theory, and the positive energy flux spectra were also obtained as exact power law solutions. The weak turbulence regime was also investigated numerically and it was shown in particular that the two-dimensional manifolds is an integrable singularity at $k_\parallel = 0$, which is related to the scaling of the energy spectrum $\propto k_\parallel^{-1/2}$, and that the energy cascade goes forward (Bellet *et al.* 2006). Recently, the problem of confinement has been addressed explicitly in the inertial wave turbulence theory using discrete wave numbers (Scott 2014): three asymptotically distinct stages in the evolution of the turbulence are found with finally a regime dominated by resonant interactions. Pseudo-spectral codes are often used to investigate numerically homogeneous rotating turbulence (see e.g. Mininni & Pouquet 2010a). Several questions have been investigated like the origin of the anisotropy or of the inverse cascade observed when a forcing is applied at intermediate scale k_f . However, according to the question addressed the results may be affected by the discretization and by finite-box effects at too small Rossby number and too long elapsed time (Smith & Lee 2005; Bourouiba 2008). This seems to be the case in particular for the question of the inverse cascade mediated by the decoupling of the slow mode. For example, it was found that the one-dimensional isotropic energy spectrum $E(k) \sim k^{-x}$ may follow two different power laws with $2 \leq x \leq 2.5$ at small-scale ($k > k_f$) and $x \simeq 3$ at large-scale ($k < k_f$)

(Smith & Waleffe 1999). But it was also shown that the scaling at large-scale was strongly influenced by the value of the aspect ratio between the parallel and the perpendicular (to Ω_0) resolution, a small aspect leading to a reduction of the number of available resonant triads, hence an alteration of the spectrum with the restoration of a $k^{-5/3}$ spectrum for small enough vertical resolution. Several experiments have been devoted to rotating turbulence with different types of apparatus (Hopfinger *et al.* 1982; Jacquin *et al.* 1990; Baroud *et al.* 2002; Morize *et al.* 2005; van Bokhoven *et al.* 2009). Contrary to the theory and the simulation, it is very challenging to reproduce experimentally the conditions of homogeneous turbulence. Nevertheless, one of the main results reported is that the rotation leads to a bi-dimensionalisation of an initial homogeneous isotropic turbulence with anisotropic spectra where energy is preferentially accumulated in the perpendicular (to Ω_0) wave numbers k_\perp . Energy spectra with $x \geq 2$ were experimentally observed (Baroud *et al.* 2002; Morize *et al.* 2005; van Bokhoven *et al.* 2009) revealing a significant discrepancy with the isotropic Kolmogorov spectrum ($x = 5/3$) for non-rotating fluids. Note that the wave number entering in the spectral measurements corresponds mainly to k_\perp . Recently, direct measurements of energy transfer have been made in the physical space by using third-order structure function (Lamriben *et al.* 2011) and an increase of anisotropy at small scales has been found in agreement with some theoretical studies (Jacquin *et al.* 1990; Galtier 2003, 2009a; Bellet *et al.* 2006). The role of kinetic helicity – which quantifies departures from mirror symmetry (Moffatt 1969) – on rotating fluids has been the subject of few studies. One reason is that it is difficult to measure the helicity production from experimental studies. The other reason is probably linked to the negligible effect of helicity on energy in non-rotating turbulence. Indeed, in this case one observes a joint constant flux cascade of energy and helicity with a $k^{-5/3}$ spectrum for both quantities (Chen *et al.* 2003a,b). But recently, several numerical simulations have demonstrated the surprising strong impact of helicity on fast rotating hydrodynamic turbulence (Mininni & Pouquet 2010a,b; Teitelbaum & Mininni 2009; Mininni *et al.* 2012) whose main properties can be summarized as follows. When the (large-scale) forcing applied to the system injects only negligible helicity, the dynamics is mainly governed by a direct energy cascade compatible with an energy spectrum $E(k_\perp) \sim k_\perp^{-5/2}$ which is precisely the weak turbulence prediction (Galtier 2003). However, when the helicity injection becomes so important that the dynamics is mainly governed by a direct helicity cascade, different scalings are found following the empirical law:

$$n + \tilde{n} = -4, \quad (1.2)$$

where n and \tilde{n} are respectively the power law indices of the one-dimensional energy and helicity spectra. This law cannot be explained by a consistent phenomenology where anisotropy is used which renders the relation (1.2) highly non-trivial. As shown by Galtier (2014), an explanation can only be found when a rigorous analysis is made on the weak turbulence equations: the relation corresponds in fact to the finite helicity flux spectra which are exact solutions of the equations.

It has been long recognized that the Earth's magnetic field is not steady (Finlay *et al.* 2010). Changes occur across a wide range of timescales from second – because of the interactions between the solar wind and the magnetosphere – to several tens of millions years which is the longest timespan between polarity reversals. To understand the generation and the maintain of a large-scale magnetic field, the most promising mechanism is the dynamo (Pouquet *et al.* 1976; Moffatt 1978; Brandenburg 2001). Dynamo is an active area of research where dramatic developments have been made in the past several years (Dormy *et al.* 2000). The subject concerns primarily the Earth where a large

amount of data is available which allows us to follow e.g. the geomagnetic polarity reversal occurrences over million years (Finlay & Jackson 2003; Roberts & King 2013). This chaotic behavior contrasts drastically with the surprisingly regularity of the Sun which changes its magnetic field lines polarity every ~ 11 years. It is believed that the three main ingredients for the geodynamo problem are the Coriolis, Lorentz-Laplace and buoyancy forces. The latter force may be seen as a source of turbulence for the conducting fluids described by incompressible magnetohydrodynamics (MHD), whereas the two others are more or less balanced (Elsässer number of order one). This balance leads to the strong-field regime – the so-called magnetostrophic dynamo – for which we may derive magnetostrophic waves (Lehnert 1954; Schmitt *et al.* 2008). This regime is thought to be relevant not only for Earth but also for giant planets like Jupiter or Saturn, and by extension probably to exoplanets (Stevenson 2003). In order to investigate the dynamo problem several experiments have been developed (Pétrélis *et al.* 2007). In one of them, the authors were able to successfully reproduce with liquid sodium reversals and excursions of a turbulent dynamo generated by two (counter) rotating disks (Berhanu *et al.* 2007). This result follows a three-dimensional numerical simulation of the Earth’s outer core where the reversal of the dipole moment was also obtained (Glatzmaier & Roberts 1995). In this model, however, the inertial/advection terms are simply discarded to mimic a very small Rossby number. This assumption is in apparent contradiction with any turbulent regime (Reynolds number is about 10^9 for the Earth’s outer core) and in particular with the weak turbulence one in which the nonlinear interactions – although weak at short-time scales compared with the linear contributions – become important for the nonlinear dynamics at asymptotically large-time scales. As we will see below, it is basically the regime that we shall investigate theoretically in this paper: a sea of helical (magnetized) waves (Moffatt 1970) will be considered as the main ingredient for the triggering of dynamo through the nonlinear transfer of magnetic energy and helicity.

Weak turbulence is the study of the long time statistical behavior of a sea of weakly nonlinear dispersive waves (Nazarenko 2011). The energy transfer between waves occurs mostly among resonant sets of waves and the resulting energy distribution, far from a thermodynamic equilibrium, is characterized by a wide power law spectrum and a high Reynolds number. This range of wavenumbers – the inertial range – is generally localized between large-scales at which energy is injected in the system (sources) and small-scales at which waves break or dissipate (sinks). Pioneering works on weak turbulence date back to the sixties when it was established that the stochastic initial value problem for weakly coupled wave systems has a natural asymptotic closure induced by the dispersive nature of the waves and the large separation of linear and nonlinear time scales (Benney & Saffman 1966; Benney & Newell 1967, 1969). In the meantime, Zakharov & Filonenko (1966) showed that the kinetic equations derived from the weak turbulence analysis have exact equilibrium solutions which are the thermodynamic zero flux solutions but also – and more importantly – finite flux solutions which describe the transfer of conserved quantities between sources and sinks. The solutions, first published for isotropic turbulence (Zakharov 1965; Zakharov & Filonenko 1966) were then extended to anisotropic turbulence (Kuznetsov 1972). Weak turbulence is a very common natural regime with applications, for example, to capillary waves (Kolmakov *et al.* 2004), gravity waves (Falcon *et al.* 2007), superfluid helium and processes of Bose-Einstein condensation (Lvov *et al.* 2003), nonlinear optics (Dyachenko *et al.* 1992), inertial waves (Galtier 2003), Alfvén waves (Galtier *et al.* 2000, 2002; Galtier & Chandran 2006) or whistler/kinetic Alfvén waves (Galtier 2006b).

In this paper, the weak turbulence theory will be established for rotating MHD in the limit of small Rossby and Ekman numbers, the latter measuring the ratio of the viscous on

Coriolis terms. We shall assume the existence of a strong uniform magnetic field parallel to the fast and constant rotating rate. The combination of the Coriolis and Lorentz-Laplace forces leads to the appearance of two types of circularly polarized waves and a possible non equipartition between the kinetic and magnetic energies (Moffatt 1972; Favier *et al.* 2012). After a general introduction to rotating MHD in §2, a weak helical turbulence formalism is developed in §3 by using a technique developed in Galtier (2006b). The phenomenology of weak turbulence dynamo is given in §4, the general properties of the weak turbulence equations are discussed in §5, whereas the exact spectral solutions are derived in §6. We conclude with a discussion in §7. Generally speaking, it is believed that the present work can be useful for better understanding the nonlinear magnetostrophic dynamo (Roberts & King 2013) with in background an application to Earth but also to giant planets like Jupiter or Saturn for which the intensity of the Coriolis force is relatively strong, the range of available length scales wide, and the magnetic field mainly dipolar with a weak tilt ($\leq 10^\circ$) of the dipole relative to the rotation axis. By extension we may even think that the analysis is relevant for exoplanets and some magnetized stars (Morin *et al.* 2011).

2. Rotating magnetohydrodynamics

2.1. Governing equations

The basic equations governing incompressible MHD under solid rotation and in the presence of a uniform background magnetic field are:

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b}_0 \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \nabla^2 \mathbf{u}, \quad (2.1)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b}_0 \cdot \nabla \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b}, \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.3)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (2.4)$$

with \mathbf{u} the velocity, P_* the total pressure (including the magnetic pressure and the centrifugal term), \mathbf{b} the magnetic field normalized to a velocity ($\mathbf{b} \rightarrow \sqrt{\mu_0 \rho_0} \mathbf{b}$, with ρ_0 the constant density), \mathbf{b}_0 the uniform normalized magnetic field, Ω_0 the rotating rate, ν the kinematic viscosity and η the magnetic diffusivity. Note the presence of the Coriolis force in the first equation (second term in the left hand side). Turbulence can only be maintained if a source is added to balance the small-scale dissipation. For example, in the geodynamo problem we may think that this external forcing is played by the convection (since the Rayleigh number $\sim 10^9$) with the buoyancy force (Braginsky & Roberts 1995). In our case, we shall perform a pure nonlinear analysis, therefore the source and dissipation terms will be discarded. The weak turbulence equations that will be derived may describe, however, *any* magnetic Prandtl limit since the (linear) dissipative terms may be added to the equations after having made the nonlinear asymptotic analysis. In the rest of the paper, we shall assume that:

$$\boldsymbol{\Omega}_0 = \Omega_0 \hat{\mathbf{e}}_{\parallel}, \quad \mathbf{b}_0 = b_0 \hat{\mathbf{e}}_{\parallel}, \quad (2.5)$$

with $\hat{\mathbf{e}}_{\parallel}$ a unit vector ($|\hat{\mathbf{e}}_{\parallel}| = 1$). We introduce the magneto-inertial length d defined as:

$$d \equiv \frac{b_0}{\Omega_0}. \quad (2.6)$$

This length scale will be useful to characterize the main properties of rotating MHD.

2.2. Three-dimensional inviscid invariants

The two inviscid ($\nu = \eta = 0$) quadratic invariants of incompressible rotating MHD in the presence of a background magnetic field parallel to the rotating axis are the total energy:

$$E = \frac{1}{2} \int (\mathbf{u}^2 + \mathbf{b}^2) d\mathcal{V}, \quad (2.7)$$

and the hybrid helicity:

$$H = \frac{1}{2} \int \left(\mathbf{u} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{d} \right) d\mathcal{V}, \quad (2.8)$$

where \mathbf{a} is the vector potential ($\mathbf{b} = \nabla \times \mathbf{a}$) and \mathcal{V} is the volume over which the average is made. The second invariant is a mixture of cross-helicity, $H^c = (1/2) \int (\mathbf{u} \cdot \mathbf{b}) d\mathcal{V}$, and magnetic helicity, $H^m = (1/2) \int (\mathbf{a} \cdot \mathbf{b}) d\mathcal{V}$, which are not conserved in the present situation (Matthaeus & Goldstein 1982). Indeed, it is straightforward to show from (2.1)–(2.4) that (see also Shebalin 2006):

$$\frac{\partial E}{\partial t} = - \int (\nu \mathbf{w}^2 + \eta \mathbf{j}^2) d\mathcal{V}, \quad (2.9)$$

$$\frac{\partial H^c}{\partial t} = \boldsymbol{\Omega}_0 \cdot \int (\mathbf{b} \times \mathbf{u}) d\mathcal{V} - (\nu + \eta) \int (\mathbf{j} \cdot \mathbf{w}) d\mathcal{V}, \quad (2.10)$$

$$\frac{\partial H^m}{\partial t} = \mathbf{b}_0 \cdot \int (\mathbf{b} \times \mathbf{u}) d\mathcal{V} - 2\eta \int (\mathbf{j} \cdot \mathbf{b}) d\mathcal{V}, \quad (2.11)$$

where \mathbf{w} is the vorticity and \mathbf{j} is the normalized current density. Therefore, the previous equations demonstrate that a second invariant may emerge *if and only if* $\mathbf{b}_0 = d\boldsymbol{\Omega}_0$. Below, we will verify that for the weak turbulence equations these two inviscid invariants are conserved for each triad of wave vectors.

2.3. Helical MHD waves

One of the main effects produced by the Coriolis force is to modify the polarization of the linearly polarized Alfvén waves – solutions of the standard MHD equations – which become circularly polarized and dispersive (Lehnert 1954). Indeed, if we linearize equations (2.1)–(2.4) such that:

$$\mathbf{b}(\mathbf{x}) = \epsilon \mathbf{b}(\mathbf{x}), \quad \mathbf{u}(\mathbf{x}) = \epsilon \mathbf{u}(\mathbf{x}), \quad (2.12)$$

with ϵ a small parameter ($0 < \epsilon \ll 1$) and \mathbf{x} a three-dimensional displacement vector, then we obtain the following inviscid ($\nu = 0$) and ideal ($\eta = 0$) equations in Fourier space:

$$\partial_t \mathbf{w}_{\mathbf{k}} - 2ik_{\parallel} \Omega_0 \mathbf{u}_{\mathbf{k}} - ik_{\parallel} b_0 \mathbf{j}_{\mathbf{k}} = \epsilon \{ \mathbf{w} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{b} \cdot \nabla \mathbf{j} - \mathbf{j} \cdot \nabla \mathbf{b} \}_{\mathbf{k}}, \quad (2.13)$$

$$\partial_t \mathbf{b}_{\mathbf{k}} - ik_{\parallel} b_0 \mathbf{u}_{\mathbf{k}} = \epsilon \{ \mathbf{b} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{b} \}_{\mathbf{k}}, \quad (2.14)$$

$$\mathbf{k} \cdot \mathbf{u}_{\mathbf{k}} = 0, \quad (2.15)$$

$$\mathbf{k} \cdot \mathbf{b}_{\mathbf{k}} = 0, \quad (2.16)$$

where the wave vector $\mathbf{k} = k \hat{\mathbf{e}}_k = \mathbf{k}_{\perp} + k_{\parallel} \hat{\mathbf{e}}_{\parallel}$ ($k = |\mathbf{k}|$, $k_{\perp} = |\mathbf{k}_{\perp}|$, $|\hat{\mathbf{e}}_k| = 1$) and $i^2 = -1$. The index \mathbf{k} denotes the Fourier transform, defined by the relation:

$$\mathbf{u}(\mathbf{x}) \equiv \int \mathbf{u}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad (2.17)$$

where $\mathbf{u}(\mathbf{k}) = \mathbf{u}_{\mathbf{k}} = \tilde{\mathbf{u}}_{\mathbf{k}} e^{-i\omega t}$ (the same notation is used for the other fields). The linear dispersion relation ($\epsilon = 0$) reads:

$$\omega^2 + \left(\frac{2\Omega_0 k_{\parallel}}{\Lambda k} \right) \omega - k_{\parallel}^2 b_0^2 = 0, \quad (2.18)$$

with:

$$\begin{Bmatrix} \tilde{\mathbf{u}}_{\mathbf{k}} \\ \tilde{\mathbf{b}}_{\mathbf{k}} \end{Bmatrix} = \Lambda i \hat{\mathbf{e}}_k \times \begin{Bmatrix} \tilde{\mathbf{u}}_{\mathbf{k}} \\ \tilde{\mathbf{b}}_{\mathbf{k}} \end{Bmatrix}. \quad (2.19)$$

We obtain the general solution:

$$\omega \equiv \omega_{\Lambda}^s = \frac{sk_{\parallel}\Omega_0}{k} \left(-s\Lambda + \sqrt{1 + k^2 d^2} \right), \quad (2.20)$$

where the value (± 1) of s defines the directional wave polarity such that we always have $sk_{\parallel} \geq 0$; then ω_{Λ}^s is a positive definite pulsation. The wave polarization Λ tells us if the wave is right ($\Lambda = s$) or left ($\Lambda = -s$) circularly polarized. In the first case, we are dealing with the magnetostrophic branch, whereas in the latter case with the inertial branch (see figure 1). We see that the transverse circularly polarized waves are dispersive and that we recover the two well-known limits, *i.e.* the pure inertial waves ($\omega_{-s}^s = 2s\Omega_0 k_{\parallel}/k \equiv \omega_I$) in the large-scale limit ($kd \rightarrow 0$), and the standard Alfvén waves ($\omega = sk_{\parallel} b_0 \equiv \omega_A$) in the small-scale limit ($kd \rightarrow +\infty$). For the pure magnetostrophic waves we find the pulsation, $\omega_s^s = sk_{\parallel} kdb_0/2 = \omega_A^2/\omega_I \equiv \omega_M$. Note that the Alfvén waves become linearly polarized only when the Coriolis force vanishes: when it is present, whatever its magnitude is, the modified Alfvén waves are circularly polarized. This property is also found in MHD when the Hall term is added (Sahraoui *et al.* 2007).

2.4. Polarization

The polarizations s and Λ can be related to two well-known quantities, the reduced magnetic helicity σ^m and the reduced cross-helicity σ^c . The reduced magnetic helicity is defined as:

$$\sigma^m = \frac{\mathbf{a}_{\mathbf{k}} \cdot \mathbf{b}_{\mathbf{k}}^* + \mathbf{a}_{\mathbf{k}}^* \cdot \mathbf{b}_{\mathbf{k}}}{2|\mathbf{a}_{\mathbf{k}}||\mathbf{b}_{\mathbf{k}}|}, \quad (2.21)$$

where $*$ denotes the complex conjugate. For circularly polarized waves, we can use relation (2.19) which gives $\sigma^m = \Lambda$. On the other hand, the reduced cross-helicity is defined as:

$$\sigma^c = \frac{\mathbf{u}_{\mathbf{k}} \cdot \mathbf{b}_{\mathbf{k}}^* + \mathbf{u}_{\mathbf{k}}^* \cdot \mathbf{b}_{\mathbf{k}}}{2|\mathbf{u}_{\mathbf{k}}||\mathbf{b}_{\mathbf{k}}|}. \quad (2.22)$$

The linear solution implies, $\omega \mathbf{b}_{\mathbf{k}} = -s|k_{\parallel}|b_0 \mathbf{u}_{\mathbf{k}}$, which leads to $\sigma^c = -s$. The use of both relations gives eventually:

$$\sigma^m \sigma^c = -\Lambda s. \quad (2.23)$$

This result is only valid for the linear solutions but may be generalized to any fluctuations in order to find the properties of helical turbulence (Meyrand & Galtier 2012).

2.5. Magnetostrophic equation

The governing equations of rotating MHD can also be written in the following form:

$$\frac{\partial \mathbf{w}}{\partial t} = \nabla \times [\mathbf{u} \times (\mathbf{w} + 2\Omega_0 \mathbf{e}_0) + \mathbf{j} \times (\mathbf{b} + d\Omega_0 \mathbf{e}_0)] + \nu \nabla^2 \mathbf{w}, \quad (2.24)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [\mathbf{u} \times (\mathbf{b} + \mathbf{b}_0)] + \eta \nabla^2 \mathbf{b}, \quad (2.25)$$

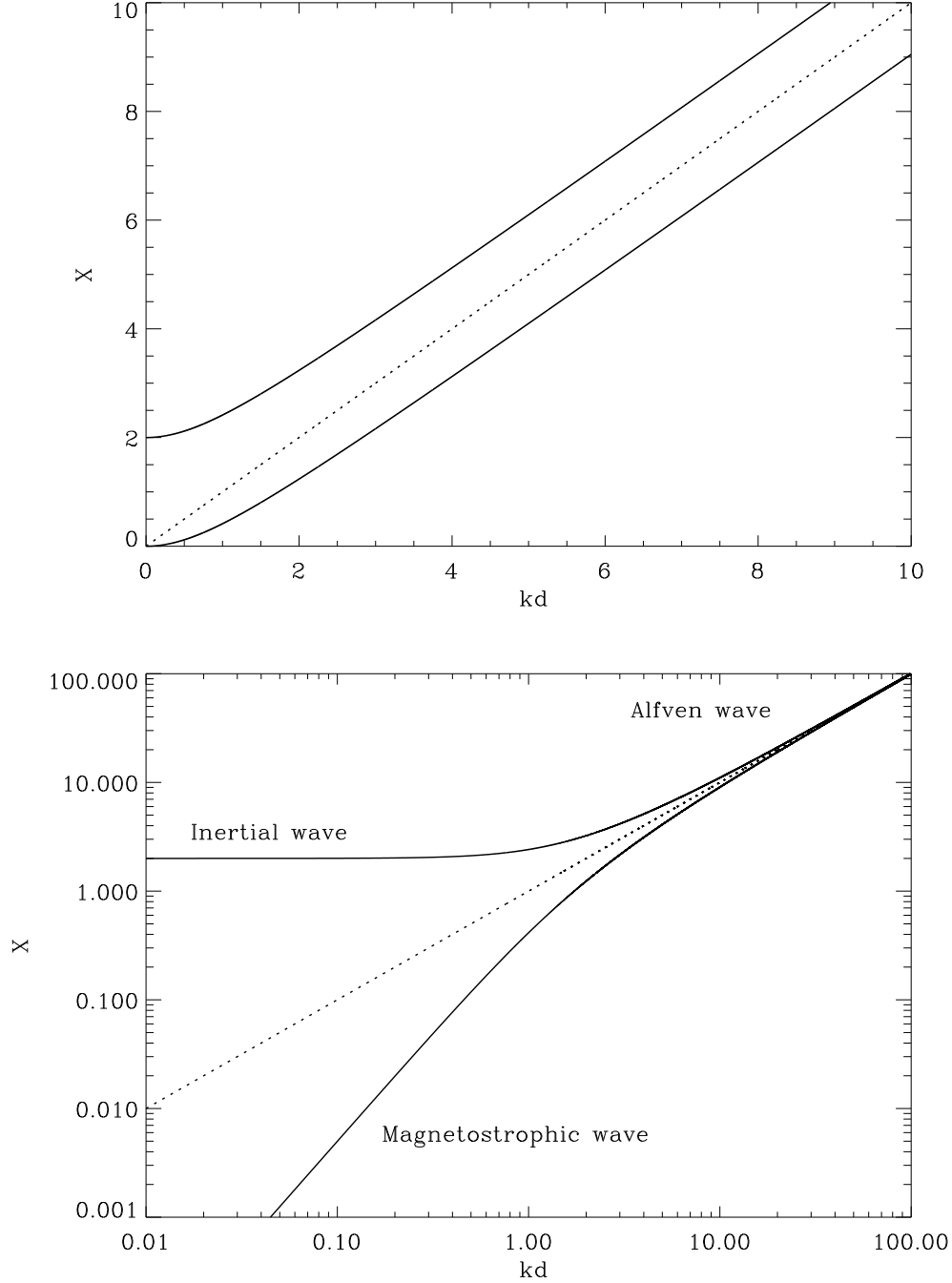


FIGURE 1. Dispersion relation for rotating MHD permeated by a background magnetic field in linear (top) and logarithmic (bottom) coordinates, with $X \equiv k\omega_A^2/(sk_{\parallel}\Omega_0)$. The upper and lower branches correspond respectively to left- and right-handed polarized waves. The Alfvén wave dispersion relation is also given (dotted line).

where the relation $\mathbf{b}_0 = d\boldsymbol{\Omega}_0$ has been introduced. The magnetostrophic regime corresponds to a balance between the Coriolis and the Lorentz-Laplace forces (Finlay 2008). If we balance such terms in the linear case, we obtain the relation:

$$2\mathbf{u} = -d\mathbf{j}, \quad (2.26)$$

which can be introduced in equation (2.25) to give:

$$\frac{\partial \mathbf{b}}{\partial t} = -\frac{d}{2} \nabla \times [(\nabla \times \mathbf{b}) \times (\mathbf{b} + \mathbf{b}_0)] + \eta \nabla^2 \mathbf{b}. \quad (2.27)$$

Expression (2.27) is the magnetostrophic equation which describes the nonlinear evolution of the magnetic field when both the rotation and the uniform magnetic field are relatively strong. It is asymptotically true in the sense that it only corresponds to the lower part of the magnetostrophic branch shown in figure 1. We may note immediately the similarity with the electron MHD equation introduced in plasma physics (Kingsep *et al.* 1990) to describe the small space-time evolution of a magnetized plasma. The difference resides in the coefficient $d/2$ which is the ion skin depth d_i in electron MHD. Then, it is not surprising that the linear solution gives the same (up to a factor 1/2) dispersion relation as for whistler waves which are also right circularly polarized. We will see in section 5.6 that indeed the general weak turbulence equations gives in the large-scale right-polarization limit the same equation (up to a factor) as in the electron MHD case (Galtier & Bhattacharjee 2003).

2.6. Complex helicity decomposition

Given the incompressibility constraints (2.15) and (2.16), it is convenient to project the rotating MHD equations in a plane orthogonal to \mathbf{k} . We will use the complex helicity decomposition technique which has been shown to be effective in providing a compact description of the dynamics of three-dimensional incompressible fluids (Craya 1954; Kraichnan 1973; Waleffe 1992; Lesieur 1997; Turner 2000; Galtier 2003, 2006*b*). The complex helicity basis is also particularly useful since it allows to diagonalize systems dealing with circularly polarized waves. We introduce the complex helicity decomposition:

$$\mathbf{h}^\Lambda(\mathbf{k}) \equiv \mathbf{h}_\mathbf{k}^\Lambda = \hat{\mathbf{e}}_\theta + i\Lambda \hat{\mathbf{e}}_\Phi, \quad (2.28)$$

where:

$$\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\Phi \times \hat{\mathbf{e}}_k, \quad \hat{\mathbf{e}}_\Phi = \frac{\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k}{|\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k|}, \quad (2.29)$$

and $|\hat{\mathbf{e}}_\theta(\mathbf{k})| = |\hat{\mathbf{e}}_\Phi(\mathbf{k})| = 1$. We note that $(\hat{\mathbf{e}}_k, h_\mathbf{k}^+, h_\mathbf{k}^-)$ form a complex basis with the following properties:

$$\mathbf{h}_\mathbf{k}^{-\Lambda} = \mathbf{h}_{-\mathbf{k}}^\Lambda, \quad (2.30)$$

$$\hat{\mathbf{e}}_k \times \mathbf{h}_\mathbf{k}^\Lambda = -i\Lambda \mathbf{h}_\mathbf{k}^\Lambda, \quad (2.31)$$

$$\mathbf{k} \cdot \mathbf{h}_\mathbf{k}^\Lambda = 0, \quad (2.32)$$

$$\mathbf{h}_\mathbf{k}^\Lambda \cdot \mathbf{h}_\mathbf{k}^{\Lambda'} = 2\delta_{-\Lambda'\Lambda}. \quad (2.33)$$

We project the Fourier transform of the original vectors $\mathbf{u}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ on the helicity basis (see also Appendix B):

$$\mathbf{u}_\mathbf{k} = \sum_\Lambda \mathcal{U}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda = \sum_\Lambda \mathcal{U}_\Lambda \mathbf{h}_\mathbf{k}^\Lambda, \quad (2.34)$$

$$\mathbf{b}_\mathbf{k} = \sum_\Lambda \mathcal{B}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda = \sum_\Lambda \mathcal{B}_\Lambda \mathbf{h}_\mathbf{k}^\Lambda, \quad (2.35)$$

and in particular, we note that:

$$\mathbf{w}_{\mathbf{k}} = k \sum_{\Lambda} \Lambda \mathcal{U}_{\Lambda} \mathbf{h}_{\mathbf{k}}^{\Lambda}, \quad (2.36)$$

$$\mathbf{j}_{\mathbf{k}} = k \sum_{\Lambda} \Lambda \mathcal{B}_{\Lambda} \mathbf{h}_{\mathbf{k}}^{\Lambda}. \quad (2.37)$$

We introduce the expressions of the new fields into the rotating MHD equations written in Fourier space and we multiply it by the vector $\mathbf{h}_{-\mathbf{k}}^{\Lambda}$. First, we will focus on the linear dispersion relation ($\epsilon = 0$) which reads:

$$\partial_t \mathcal{Z}_{\Lambda}^s = -i \omega_{\Lambda}^s \mathcal{Z}_{\Lambda}^s, \quad (2.38)$$

with:

$$\mathcal{Z}_{\Lambda}^s \equiv \mathcal{U}_{\Lambda} + \xi_{\Lambda}^s \mathcal{B}_{\Lambda}, \quad (2.39)$$

$$\xi_{\Lambda}^s \equiv \frac{-skd}{(-s\Lambda + \sqrt{1 + k^2 d^2})}. \quad (2.40)$$

Equation (2.38) shows that \mathcal{Z}_{Λ}^s are the canonical variables for our system. These eigenvectors combine the velocity and the magnetic field in a non trivial way by a factor ξ_{Λ}^s (with $\omega_{\Lambda}^s = -b_0 k_{\parallel} / \xi_{\Lambda}^s$). In the small-scale limit ($kd \rightarrow +\infty$), we see that $\xi_{\Lambda}^s \rightarrow -s$: the Elsässer variables used in standard MHD are then recovered. In the large-scale limit ($kd \rightarrow 0$), we have $\xi_{\Lambda}^s \rightarrow -s kd/2$ for $\Lambda = -s$ (inertial waves), or $\xi_{\Lambda}^s \rightarrow (-2s/kd)^{-1}$ for $\Lambda = s$ (magnetostrophic wave). Therefore, \mathcal{Z}_{Λ}^s can be seen as a generalization of the Elsässer variables to rotating MHD. In the rest of the paper, we shall use the relation:

$$\mathcal{Z}_{\Lambda}^s = (\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}) a_{\Lambda}^s e^{-i\omega_{\Lambda}^s t}, \quad (2.41)$$

where a_{Λ}^s is the wave amplitude in the interaction representation for which we have, in the linear approximation, $\partial_t a_{\Lambda}^s = 0$. In particular, that means that weak nonlinearities will modify only slowly in time the helical MHD wave amplitudes. The coefficient in front of the wave amplitude is introduced in advance to simplify the algebra that we are going to develop.

3. Helical weak turbulence formalism

3.1. Fundamental equations

We decompose the inviscid nonlinear MHD equations (2.13)–(2.14) on the complex helicity basis introduced in the previous section. Then, we project the equations on the vector $\mathbf{h}_{-\mathbf{k}}^{\Lambda}$. After simplifications we obtain:

$$\partial_t \mathcal{U}_{\Lambda} - \frac{2i\Lambda\Omega_0 k_{\parallel}}{k} \mathcal{U}_{\Lambda} - ib_0 k_{\parallel} \mathcal{B}_{\Lambda} = \quad (3.1)$$

$$\frac{i\epsilon}{2\Lambda k} \int \sum_{\Lambda_p, \Lambda_q} (p\Lambda_p - q\Lambda_q) (\mathcal{U}_{\Lambda_p} \mathcal{U}_{\Lambda_q} - \mathcal{B}_{\Lambda_p} \mathcal{B}_{\Lambda_q}) (\mathbf{q} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) \delta_{pq,k} d\mathbf{p} d\mathbf{q},$$

and:

$$\partial_t \mathcal{B}_{\Lambda} - ib_0 k_{\parallel} \mathcal{U}_{\Lambda} = \quad (3.2)$$

$$\frac{i\epsilon}{2} \int \sum_{\Lambda_p, \Lambda_q} (\mathcal{U}_{\Lambda_q} \mathcal{B}_{\Lambda_p} - \mathcal{U}_{\Lambda_p} \mathcal{B}_{\Lambda_q}) (\mathbf{q} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) \delta_{pq,k} d\mathbf{p} d\mathbf{q},$$

where $\delta_{pq,k} = \delta(\mathbf{p} + \mathbf{q} - \mathbf{k})$. The delta distributions come from the Fourier transforms of the nonlinear terms. We introduce the generalized Elsässer variables in the following way:

$$\mathcal{U}_\Lambda = \sum_s \frac{-\xi_\Lambda^{-s} \mathcal{Z}_\Lambda^s}{\xi_\Lambda^s - \xi_\Lambda^{-s}}, \quad (3.3)$$

$$\mathcal{B}_\Lambda = \sum_s \frac{\mathcal{Z}_\Lambda^s}{\xi_\Lambda^s - \xi_\Lambda^{-s}}. \quad (3.4)$$

Then, we obtain in the interaction representation (variable a_Λ^s):

$$\partial_t a_\Lambda^s = \frac{i\epsilon}{2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} L_{-k \, p \, q}^{\Lambda_p \Lambda_q \, s_p s_q} a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} e^{-i\Omega_{pq,k} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \quad (3.5)$$

where:

$$L_{-k \, p \, q}^{\Lambda_p \Lambda_q \, s_p s_q} = \quad (3.6)$$

$$\left[\left(\frac{p\Lambda_p - q\Lambda_q}{\Lambda k} \right) (\xi_{\Lambda_p}^{-s_p} \xi_{\Lambda_q}^{-s_q} - 1) + \xi_\Lambda^s (\xi_{\Lambda_p}^{-s_p} - \xi_{\Lambda_q}^{-s_q}) \right] \frac{(\mathbf{q} \cdot \mathbf{h}_p^{\Lambda_p}) (\mathbf{h}_q^{\Lambda_q} \cdot \mathbf{h}_k^{\Lambda_k})}{\xi_\Lambda^s - \xi_\Lambda^{-s}},$$

and:

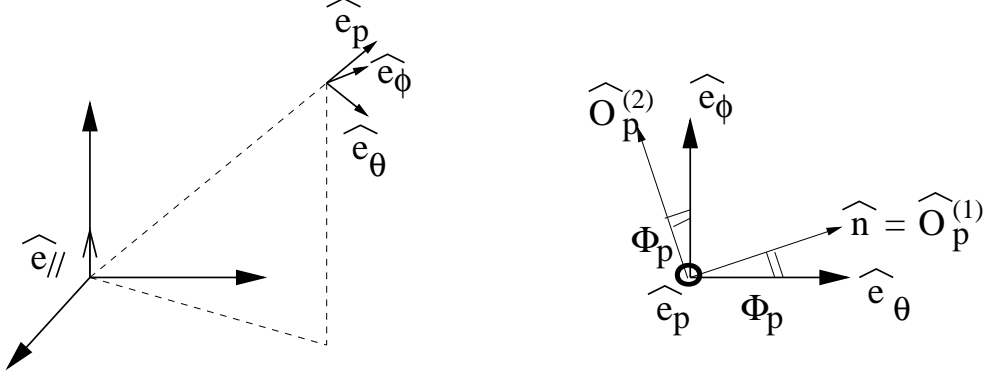
$$\Omega_{pq,k} = \omega_{\Lambda_p}^{s_p} + \omega_{\Lambda_q}^{s_q} - \omega_\Lambda^s. \quad (3.7)$$

Equation (3.5) is the wave amplitude equation from which it is possible to extract some information. As expected we see that the nonlinear terms are of order ϵ . This means that weak nonlinearities will modify only slowly in time the helical MHD wave amplitude. They contain an exponentially oscillating term which is essential for the asymptotic closure. Indeed, weak turbulence deals with variations of spectral densities at very large time, *i.e.* for a nonlinear transfer time much greater than the wave period. As a consequence, most of the nonlinear terms are destroyed by phase mixing and only a few of them, the resonance terms, survive (see e.g. Newell *et al.* (2001)). The expression obtained for the fundamental equation (3.5) is classical in weak turbulence. The main difference between different problems is localized in the matrix L which is interpreted as a complex geometric coefficient. We will see below that the local decomposition allows to get a polar form for such a coefficient which is much easier to manipulate. From equation (3.5) we see eventually that, contrary to incompressible MHD, there is no exact solutions to the nonlinear problem in incompressible rotating MHD. The origin of such a difference is that in MHD the nonlinear term involves Alfvén waves traveling only in opposite directions whereas in rotating MHD this constrain does not exist (we have a summation over Λ and s). In other words, if one type of wave is not present in incompressible MHD then the nonlinear term cancels whereas in the present problem it is not the case (see e.g. Galtier *et al.* 2000).

3.2. Local decomposition

In order to evaluate the scalar products of complex helical vectors found in the geometric coefficient (3.6), it is convenient to introduce a vector basis local to each particular triad (Waleffe 1992; Turner 2000; Galtier 2003). For example, for a given vector \mathbf{p} , we define the orthonormal basis vectors:

$$\hat{\mathbf{O}}^{(1)}(\mathbf{p}) = \hat{\mathbf{n}}, \quad (3.8)$$

FIGURE 2. Local decomposition for a given wave vector \mathbf{p} .

$$\hat{\mathbf{O}}^{(2)}(\mathbf{p}) = \hat{\mathbf{e}}_p \times \hat{\mathbf{n}}, \quad (3.9)$$

$$\hat{\mathbf{O}}^{(3)}(\mathbf{p}) = \hat{\mathbf{e}}_p, \quad (3.10)$$

where $\hat{\mathbf{e}}_p = \mathbf{p}/|\mathbf{p}|$ and:

$$\hat{\mathbf{n}} = \frac{\mathbf{p} \times \mathbf{k}}{|\mathbf{p} \times \mathbf{k}|} = \frac{\mathbf{q} \times \mathbf{p}}{|\mathbf{q} \times \mathbf{p}|} = \frac{\mathbf{k} \times \mathbf{q}}{|\mathbf{k} \times \mathbf{q}|}. \quad (3.11)$$

We see that the vector $\hat{\mathbf{n}}$ is normal to any vector of the triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ and changes sign if \mathbf{p} and \mathbf{q} are interchanged, *i.e.* $\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{q}, \mathbf{p})} = -\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{p}, \mathbf{q})}$. Note that $\hat{\mathbf{n}}$ does not change by cyclic permutation, *i.e.* $\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{q}, \mathbf{p})} = \hat{\mathbf{n}}_{(\mathbf{q}, \mathbf{p}, \mathbf{k})} = \hat{\mathbf{n}}_{(\mathbf{p}, \mathbf{k}, \mathbf{q})}$. A sketch of the local decomposition is given in figure 2. We now introduce the vectors:

$$\Xi^{\Lambda_p}(\mathbf{p}) \equiv \Xi_{\mathbf{p}}^{\Lambda_p} = \hat{\mathbf{O}}^{(1)}(\mathbf{p}) + i\Lambda_p \hat{\mathbf{O}}^{(2)}(\mathbf{p}), \quad (3.12)$$

and define the rotation angle Φ_p , so that:

$$\cos \Phi_p = \hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_\theta(\mathbf{p}), \quad (3.13)$$

$$\sin \Phi_p = \hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_\phi(\mathbf{p}). \quad (3.14)$$

The decomposition of the helicity vector $\mathbf{h}_{\mathbf{p}}^{\Lambda_p}$ in the local basis gives (similar forms are obtained for \mathbf{k} and \mathbf{q}):

$$\mathbf{h}_{\mathbf{p}}^{\Lambda_p} = \Xi_{\mathbf{p}}^{\Lambda_p} e^{i\Lambda_p \Phi_p}. \quad (3.15)$$

After some algebra we obtain the following polar form for the matrix L :

$$L_{\substack{\Lambda_p \Lambda_q \\ s_p s_q \\ k p q}} = - \left[\left(\frac{p\Lambda_p - q\Lambda_q}{\Lambda k} \right) \left(\xi_{\Lambda_p}^{-s_p} \xi_{\Lambda_q}^{-s_q} - 1 \right) + \xi_{\Lambda}^s \left(\xi_{\Lambda_p}^{-s_p} - \xi_{\Lambda_q}^{-s_q} \right) \right] \quad (3.16)$$

$$i e^{i(\Lambda \Phi_k + \Lambda_p \Phi_p + \Lambda_q \Phi_q)} \frac{\Lambda \Lambda_p \Lambda_q}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \frac{\sin \psi_k}{k} k q (\Lambda \Lambda_q + \cos \psi_p).$$

The angle ψ_k refers to the angle opposite to \mathbf{k} in the triangle defined by $\mathbf{k} = \mathbf{p} + \mathbf{q}$ ($\sin \psi_k = \hat{\mathbf{n}} \cdot (\mathbf{q} \times \mathbf{p}) / |(\mathbf{q} \times \mathbf{p})|$). To obtain equation (3.16), we have also used the well-known triangle relations:

$$\frac{\sin \psi_k}{k} = \frac{\sin \psi_p}{p} = \frac{\sin \psi_q}{q}. \quad (3.17)$$

Further modifications have to be made before applying the spectral formalism. In particular, the fundamental equation has to be invariant under interchange of \mathbf{p} and \mathbf{q} . To

do so, we shall introduce the symmetrized matrix:

$$\frac{1}{2} \left(L \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} + L \begin{smallmatrix} \Lambda_q \Lambda_p \\ s_q s_p \\ k q p \end{smallmatrix} \right). \quad (3.18)$$

Finally, by using the identities given in Appendix A, we obtain:

$$\partial_t a_\Lambda^s = \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ -k p q \end{smallmatrix} a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} e^{-i\Omega_{pq,k} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \quad (3.19)$$

where:

$$M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} = e^{i(\Lambda\Phi_k + \Lambda_p\Phi_p + \Lambda_q\Phi_q)} (\Lambda k + \Lambda_p p + \Lambda_q q) k p q \frac{\sin \psi_k}{k} \quad (3.20)$$

$$\xi_\Lambda^s \xi_{\Lambda_p}^{s_p} \xi_{\Lambda_q}^{s_q} \left(2 + \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_\Lambda^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right).$$

The matrix M possesses the following properties:

$$\left(M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} \right)^* = M \begin{smallmatrix} -\Lambda - \Lambda_p - \Lambda_q \\ -s - s_p - s_q \\ k p q \end{smallmatrix} = M \begin{smallmatrix} \Lambda & \Lambda_p & \Lambda_q \\ s & s_p & s_q \\ -k & -p & -q \end{smallmatrix}, \quad (3.21)$$

$$M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda_q \Lambda_p \\ s_q s_p \\ k q p \end{smallmatrix}, \quad (3.22)$$

$$M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda_q \Lambda_p \Lambda \\ s_q s_p s \\ q p k \end{smallmatrix}, \quad (3.23)$$

$$M \begin{smallmatrix} \Lambda_p \Lambda_q \\ s_p s_q \\ k p q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda_p \Lambda \Lambda_q \\ s_p s s_q \\ p k q \end{smallmatrix}. \quad (3.24)$$

Equation (3.19) is the fundamental equation that describes the slow evolution of the wave amplitudes due to the nonlinear terms of the incompressible rotating MHD equations. It is the starting point for deriving the weak turbulence equations. The local decomposition used here allows us to represent concisely complex information in an exponential function (polar form). As we will see below, it will simplify significantly the derivation of the asymptotic equations.

From equation (3.19) we note that the nonlinear coupling between helicity states associated with wave vectors, \mathbf{p} and \mathbf{q} , vanishes when the wave vectors are collinear (since then, $\sin \psi_k = 0$). This property is similar to the one found for pure rotating hydrodynamics. It seems to be a general property for helical waves (Kraichnan 1973; Waleffe 1992; Turner 2000; Galtier 2003, 2006b). Additionally, we note that the nonlinear coupling between helicity states vanishes whenever the wave numbers p and q are equal if their associated wave and directional polarities, Λ_p , Λ_q , and s_p , s_q respectively, are also equal. In the case of inertial waves, for which we have $\Lambda = -s$ (left-handed waves), this property was already observed (Galtier 2003). Here, this finding is generalized to right and left circularly polarized waves. Note that in the large-scale limit for which we recover the linearly polarized Alfvén waves, this property tends to disappear (see also section 5.4).

We are interested by the long-time behavior of the helical wave amplitudes. From the fundamental equation (3.19), we see that the nonlinear wave coupling will come from

resonant terms such that:

$$\begin{cases} \mathbf{k} = \mathbf{p} + \mathbf{q}, \\ \frac{k_{\parallel}}{\xi_{\Lambda}^s} = \frac{p_{\parallel}}{\xi_{\Lambda_p}^{s_p}} + \frac{q_{\parallel}}{\xi_{\Lambda_q}^{s_q}}. \end{cases} \quad (3.25)$$

The resonance condition may also be written:

$$\frac{\xi_{\Lambda}^{-s} - \xi_{\Lambda_p}^{-s_p}}{q_{\parallel}} = \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda}^{-s}}{p_{\parallel}} = \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_{\parallel}}. \quad (3.26)$$

As we shall see below, relations (3.26) are useful in simplifying the weak turbulence equations and demonstrating the conservation of inviscid invariants.

3.3. Asymptotic weak turbulence equations

Weak turbulence is a state of a system composed of many simultaneously excited and interacting nonlinear waves where the energy distribution, far from thermodynamic equilibrium, is characterized by a wide power law spectrum. This range of wave numbers – the inertial range – is generally localized between large-scales at which energy is injected in the system and small dissipative scales. The origin of weak turbulence dates back to the early sixties and since then many papers have been devoted to the subject (see e.g. Hasselmann (1962); Benney & Saffman (1966); Zakharov (1967); Sagdeev & Galeev (1969); Kuznetsov (1972); Zakharov *et al.* (1992); Galtier (2009b); Nazarenko (2011)). The essence of weak turbulence is the statistical study of large ensembles of weakly interacting dispersive waves *via* a systematic asymptotic expansion in powers of small nonlinearity. This technique leads finally to the derivation of kinetic equations for quantities like the energy and more generally for the (quadratic) invariants of the system under investigation. Here, we will follow the standard Eulerian formalism of weak turbulence (see e.g. Benney & Newell (1969)).

We define the density tensor $q_{\Lambda}^s(\mathbf{k})$ for an homogeneous turbulence, such that:

$$\langle a_{\Lambda}^s(\mathbf{k}) a_{\Lambda'}^{s'}(\mathbf{k}') \rangle \equiv q_{\Lambda}^s(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta_{\Lambda\Lambda'} \delta_{ss'}, \quad (3.27)$$

for which we shall write an asymptotic closure equation. The presence of the deltas $\delta_{\Lambda\Lambda'}$ and $\delta_{ss'}$ means that correlations with opposite wave or directional polarities have no long-time influence in the wave turbulence regime; the third delta distribution $\delta(\mathbf{k} + \mathbf{k}')$ is the consequence of the homogeneity assumption. Details of the derivation of the weak turbulence equations are given in Appendix C. After a lengthy calculation, we obtain the following result:

$$\begin{aligned} \partial_t q_{\Lambda}^s(\mathbf{k}) = & \quad (3.28) \\ & \frac{\pi \epsilon^2 d^4}{64 b_0^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \xi_{\Lambda}^{s^2} \xi_{\Lambda_p}^{s_p^2} \xi_{\Lambda_q}^{s_q^2} \left(\frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_{\parallel}} \right)^2 \\ & \left(2 + \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_{\Lambda}^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\omega_{\Lambda}^s}{1 + \xi_{\Lambda}^{-s^2}} \right) q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \\ & \left[\frac{\omega_{\Lambda}^s}{(1 + \xi_{\Lambda}^{-s^2}) q_{\Lambda}^s(\mathbf{k})} - \frac{\omega_{\Lambda_p}^{s_p}}{(1 + \xi_{\Lambda_p}^{-s_p^2}) q_{\Lambda_p}^{s_p}(\mathbf{p})} - \frac{\omega_{\Lambda_q}^{s_q}}{(1 + \xi_{\Lambda_q}^{-s_q^2}) q_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Equation (3.28) is the main result of the helical weak turbulence formalism. It describes

the statistical properties of weak turbulence for rotating MHD at the lowest order, *i.e.* for three-wave interactions.

4. Phenomenology of weak turbulence dynamo

Before going to the detailed analysis of the weak turbulence regime, it is important to have a simple picture in mind of the physical process that we are going to describe. According to the properties given in Section 3.2, if we assume that the nonlinear transfer is mainly driven by local interactions ($k \sim p \sim q$), then we may only consider the stochastic collisions between counter propagating waves (Iroshnikov 1964; Kraichnan 1965) of the same kind to derive the form of the energy spectra (see figure 3): in other words, a left (right) handed wave going upward will interact much stronger with another left (right) handed wave propagating downward than one going upward.

To find the transfer time and then the energy spectrum, we first need to evaluate the modification of a wave produced by one collision. Starting from the momentum equation (for simplicity we note the wave amplitude \mathcal{Z}_ℓ and we assume anisotropy with $k \sim k_\perp$):

$$\mathcal{Z}_\ell(t + \tau_1) \sim \mathcal{Z}_\ell(t) + \tau_1 \frac{\partial \mathcal{Z}_\ell}{\partial t} \sim \mathcal{Z}_\ell(t) + \tau_1 \frac{\mathcal{Z}_\ell^2}{\ell_\perp}, \quad (4.1)$$

where τ_1 is the duration of one collision; in other words, after one collision the distortion of a wave is $\Delta_1 \mathcal{Z}_\ell \sim \tau_1 \mathcal{Z}_\ell^2 / \ell_\perp$. This distortion is going to increase with time in such a way that after N stochastic collisions the cumulative effect may be evaluated like a random walk:

$$\sum_{i=1}^N \Delta_i \mathcal{Z}_\ell \sim \tau_1 \frac{\mathcal{Z}_\ell^2}{\ell_\perp} \sqrt{\frac{t}{\tau_1}}. \quad (4.2)$$

The transfer time τ_{tr} that we are looking for is the one for which the cumulative distortion is of the order of one, *i.e.* of the order of the wave itself:

$$\mathcal{Z}_\ell \sim \tau_1 \frac{\mathcal{Z}_\ell^2}{\ell_\perp} \sqrt{\frac{\tau_{tr}}{\tau_1}}, \quad (4.3)$$

then we obtain:

$$\tau_{tr} \sim \frac{1}{\tau_1} \frac{\ell_\perp^2}{\mathcal{Z}_\ell^2} \sim \frac{\tau_{NL}^2}{\tau_1}. \quad (4.4)$$

It is basically the formula that we are going to use to evaluate the energy spectra. Let us consider inertial waves for which $\tau_1 \sim 1/\omega_I$. A classical calculation, with $\varepsilon^u \sim \mathcal{Z}_\ell^2 / \tau_{tr}$, leads finally to the bi-dimensional axisymmetric kinetic energy spectrum:

$$E^u(k_\perp, k_\parallel) \sim \sqrt{\varepsilon^u \Omega_0} k_\perp^{-5/2} k_\parallel^{-1/2}, \quad (4.5)$$

which is the prediction for weak inertial wave turbulence (Galtier 2003). Note that this solution corresponds to a constant kinetic energy flux ε^u whereas a constant kinetic helicity flux may give other solutions (Galtier 2014). For magnetostrophic waves we have $\tau_1 \sim 1/\omega_M$, but a subtlety arrives because instead of the momentum equation now we use Eq. (2.27) for which the nonlinear term leads to $\tau_{NL} \sim \ell_\perp^2 / (d\mathcal{Z}_\ell)$. Then, we obtain the bi-dimensional axisymmetric magnetic energy spectrum:

$$E^b(k_\perp, k_\parallel) \sim \sqrt{\frac{\varepsilon^b b_0}{d}} k_\perp^{-5/2} k_\parallel^{-1/2}, \quad (4.6)$$

which corresponds to a constant magnetic energy flux ε^b solution.

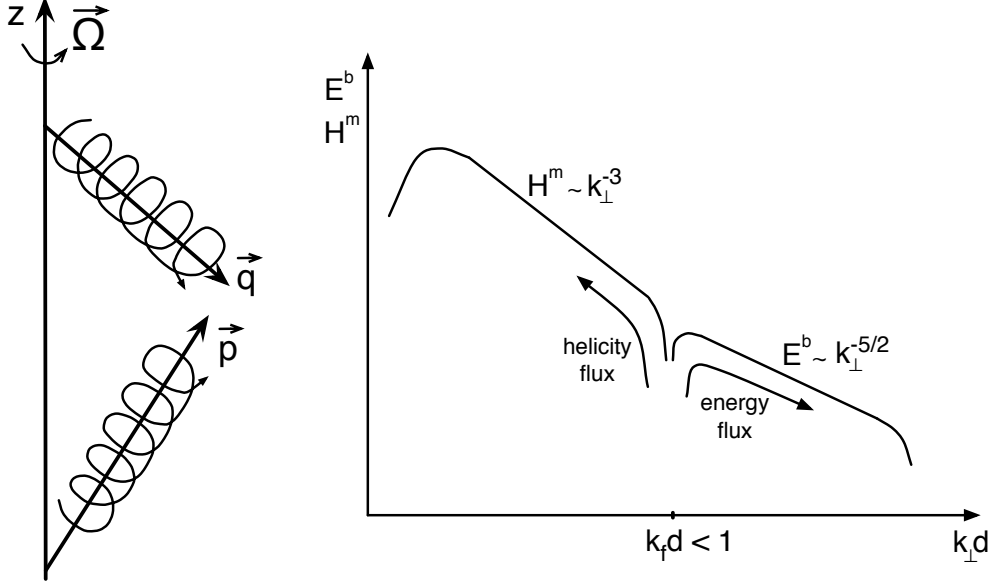


FIGURE 3. Left: collision between counter propagating circularly polarized waves. Right: heuristic view of the magnetic energy and hybrid helicity spectra with a forcing applied at an intermediate scale $k_f d < 1$; the inverse cascade of helicity may also drive the energy to the largest scales of the system.

The same heuristic analysis can be made for the other invariant, the hybrid helicity. Let us consider the most interesting case, namely the magnetostrophic regime in which the hybrid helicity is mainly dominated by the magnetic helicity (with $kd < 1$). By using the transfer time derived above (with the helicity flux $\tilde{\varepsilon} \sim H_\ell / \tau_{tr}$), we find:

$$H(k_\perp, k_\parallel) \sim H^m(k_\perp, k_\parallel) \sim \sqrt{\frac{\tilde{\varepsilon} b_0}{d^2}} k_\perp^{-3} k_\parallel^{-1/2}. \quad (4.7)$$

An inverse cascade may happen for the hybrid helicity (see figure 3) which in turn may drive the magnetic energy at the largest scales of the system. It is through this mechanism that the large-scale magnetic field can be regenerated by the weak turbulence dynamo. It is fundamental to have in mind that this cascade happens because the hybrid helicity is an inviscid and ideal invariant of rotating MHD (e.g. without rotation an inverse cascade of magnetic helicity is impossible in weak incompressible MHD turbulence (Galtier & Nazarenko 2008)). In other words, the inverse cascade should stop as soon as the mean magnetic field and the rotating rate are not collinear anymore. It is likely, however, that the inverse cascade is only weakly reduced when the mean magnetic field and the rotating rate experience a slight out of alignment (weak tilt case) and is completely inhibited in the strong tilt case. This comment might explain why the planetary magnetic fields are often dipolar with a weak tilt ($< 10^\circ$) of the dipole relative to the rotation axis. Note that the increase of the magnetic field at large-scale may lead to a state where the ratio between the magnetic and kinetic energies is significantly larger than one.

5. General properties

5.1. Basic turbulent spectra

In section 2.2, we have introduced the three-dimensional inviscid invariants of incompressible rotating MHD. The first test that the weak turbulence equations have to satisfy is the detailed conservation of these invariants, that is to say the conservation of invariants for each triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$. Starting from definitions (2.7)–(2.8), we find the total energy spectrum:

$$E(\mathbf{k}) = \sum_{\Lambda, s} (1 + \xi_{\Lambda}^{-s^2}) q_{\Lambda}^s(\mathbf{k}) \equiv \sum_{\Lambda, s} \mathcal{E}_{\Lambda}^s(\mathbf{k}), \quad (5.1)$$

which is composed of the magnetic spectrum:

$$E^b(\mathbf{k}) = \sum_{\Lambda, s} q_{\Lambda}^s(\mathbf{k}), \quad (5.2)$$

and the kinetic spectrum:

$$E^u(\mathbf{k}) = \sum_{\Lambda, s} \xi_{\Lambda}^{-s^2} q_{\Lambda}^s(\mathbf{k}). \quad (5.3)$$

We also find the cross-helicity spectrum:

$$H^c(\mathbf{k}) = - \sum_{\Lambda, s} \xi_{\Lambda}^{-s} q_{\Lambda}^s(\mathbf{k}), \quad (5.4)$$

and the magnetic helicity spectrum:

$$H^m(\mathbf{k}) = \sum_{\Lambda, s} \frac{\Lambda}{k} q_{\Lambda}^s(\mathbf{k}). \quad (5.5)$$

Note that each of these spectra may be decomposed into right ($\Lambda = s$) and left ($\Lambda = -s$) polarization spectra. From the last two expressions we find the second inviscid invariant, the hybrid helicity spectrum:

$$H(\mathbf{k}) = \sum_{\Lambda, s} \left(\frac{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}}{2} \right) q_{\Lambda}^s(\mathbf{k}) \equiv \sum_{\Lambda, s} \mathcal{H}_{\Lambda}^s(\mathbf{k}). \quad (5.6)$$

We shall demonstrate below the conservation of the energy and the hybrid helicity.

5.2. Triadic conservation of inviscid invariants

We will first check the energy conservation. From expression (3.28), we may write:

$$\begin{aligned} \partial_t E(t) &\equiv \partial_t \int E(\mathbf{k}) d\mathbf{k} \equiv \partial_t \int \sum_{\Lambda, s} \mathcal{E}_{\Lambda}^s(\mathbf{k}) d\mathbf{k} = \\ &\frac{\pi \epsilon^2 d^4}{64 b_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \xi_{\Lambda}^{s^2} \xi_{\Lambda_p}^{s_p^2} \xi_{\Lambda_q}^{s_q^2} \left(\frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_{\parallel}} \right)^2 \\ &\quad \left(2 + \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_{\Lambda}^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right)^2 q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \\ &\quad \omega_{\Lambda}^s \left[\frac{\omega_{\Lambda}^s}{\mathcal{E}_{\Lambda}^s(\mathbf{k})} + \frac{\omega_{\Lambda_p}^{s_p}}{\mathcal{E}_{\Lambda_p}^{s_p}(\mathbf{p})} + \frac{\omega_{\Lambda_q}^{s_q}}{\mathcal{E}_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(\Omega_{kpq}) \delta_{kpq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned} \quad (5.7)$$

Equation (5.7) is invariant under cyclic permutations of wave vectors; it leads to:

$$\partial_t E(t) = \quad (5.8)$$

$$\begin{aligned} & \frac{\pi \epsilon^2 d^4}{192 b_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \xi_\Lambda^{s^2} \xi_{\Lambda_p}^{s_p^2} \xi_{\Lambda_q}^{s_q^2} \left(\frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_\parallel} \right)^2 \\ & \left(2 + \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_\Lambda^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right)^2 q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \\ & \Omega_{kpq} \left[\frac{\omega_\Lambda^s}{\mathcal{E}_\Lambda^s(\mathbf{k})} + \frac{\omega_{\Lambda_p}^{s_p}}{\mathcal{E}_{\Lambda_p}^{s_p}(\mathbf{p})} + \frac{\omega_{\Lambda_q}^{s_q}}{\mathcal{E}_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(\Omega_{kpq}) \delta_{kpq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

On the resonant manifold $\Omega_{kpq} = 0$, therefore the total energy is conserved exactly for each triad: we have a detailed conservation of the total energy.

For the second invariant it is straightforward to show with relation (A 3) that:

$$\partial_t H(t) \equiv \partial_t \int H(\mathbf{k}) d\mathbf{k} \equiv \partial_t \int \sum_{\Lambda, s} \mathcal{H}_\Lambda^s(\mathbf{k}) d\mathbf{k} = \quad (5.9)$$

$$\begin{aligned} & \frac{\pi \epsilon^2 d^4}{64 b_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \xi_\Lambda^{s^2} \xi_{\Lambda_p}^{s_p^2} \xi_{\Lambda_q}^{s_q^2} \left(\frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_\parallel} \right)^2 \\ & \left(2 + \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_\Lambda^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right)^2 q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \\ & \frac{\xi_\Lambda^s}{2} \omega_\Lambda^s \left[\frac{2\xi_\Lambda^s \omega_\Lambda^s}{\mathcal{H}_\Lambda^s(\mathbf{k})} + \frac{2\xi_{\Lambda_p}^{s_p} \omega_{\Lambda_p}^{s_p}}{\mathcal{H}_{\Lambda_p}^{s_p}(\mathbf{p})} + \frac{2\xi_{\Lambda_q}^{s_q} \omega_{\Lambda_q}^{s_q}}{\mathcal{H}_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(\Omega_{kpq}) \delta_{kpq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Equation (5.7) is also invariant under cyclic permutations of wave vectors. Then, one is led to:

$$\partial_t H(t) = \quad (5.10)$$

$$\begin{aligned} & \frac{\pi \epsilon^2 d^4}{192} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \xi_\Lambda^{s^2} \xi_{\Lambda_p}^{s_p^2} \xi_{\Lambda_q}^{s_q^2} \left(\frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{k_\parallel} \right)^2 \\ & \left(2 + \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_\Lambda^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2} \right)^2 q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \\ & (k_\parallel + p_\parallel + q_\parallel) \left[\frac{k_\parallel}{\mathcal{H}_\Lambda^s(\mathbf{k})} + \frac{p_\parallel}{\mathcal{H}_{\Lambda_p}^{s_p}(\mathbf{p})} + \frac{q_\parallel}{\mathcal{H}_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(\Omega_{kpq}) \delta_{kpq} d\mathbf{k} d\mathbf{p} d\mathbf{q}, \end{aligned}$$

which is exactly equal to zero on the resonant manifold: we also have the triadic conservation for the hybrid helicity.

5.3. Helical properties

From the weak turbulence equations (3.28), we find several general properties. Some of them can be obtained directly from the wave amplitude equation (3.19) as explained in section 3.2. First, we observe that there is no coupling between helical waves associated with wave vectors, \mathbf{p} and \mathbf{q} , when the wave vectors are collinear ($\sin \psi_k = 0$). Second, we note that there is no coupling between helical waves associate with vectors \mathbf{p} and \mathbf{q}

whenever their magnitudes, p and q , are equal if their associated polarities, s_p and s_q in one hand and, Λ_p and Λ_q on the other hand, are also equal (since then $\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p} = 0$). These properties hold for the inviscid invariants and generalize what was found previously for rotating hydrodynamics (Galtier 2003) where we only have left circularly polarized waves ($\Lambda = -s$). It seems to be a generic property of helical wave interactions (Kraichnan 1973; Waleffe 1992; Turner 2000). As noted before, this property tends to disappear when the large-scale limit is taken, *i.e.* when we tend to the standard MHD. Third, it follows from the previous observations that a strong helical perturbation localized initially in a narrow band of wave numbers will lead to a weak transfer of total energy and hybrid helicities. Note that these properties can be inferred from the fundamental equation (3.19) as well.

5.4. Small-scale dynamics: Alfvén waves

We start with the general weak turbulence equation (3.28) and take the small-scale limit ($kd \rightarrow +\infty$) for which we have, at the leading order:

$$\xi_{\Lambda}^s \rightarrow -s, \quad (5.11)$$

$$\left(\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}\right)^2 \rightarrow (s_q - s_p)^2, \quad (5.12)$$

$$\left(2 + \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_{\Lambda}^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2}\right)^2 \rightarrow \frac{16}{d^4} \left(\frac{s\Lambda k + s_p\Lambda_p p + s_q\Lambda_q q}{kpq}\right)^2, \quad (5.13)$$

$$\omega_{\Lambda}^s \rightarrow sk_{\parallel} b_0 = \omega_A. \quad (5.14)$$

After introducing the previous expressions into (3.28), we obtain:

$$\partial_t q_{\Lambda}^s(\mathbf{k}) = \quad (5.15)$$

$$\frac{\pi \epsilon^2}{16 b_0} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \left(\frac{\sin \psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(\frac{s_q - s_p}{k_{\parallel}}\right)^2 (s\Lambda k + s_p\Lambda_p p + s_q\Lambda_q q)^2 sk_{\parallel} \\ q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \left[\frac{sk_{\parallel}}{q_{\Lambda}^s(\mathbf{k})} - \frac{s_p p_{\parallel}}{q_{\Lambda_p}^{s_p}(\mathbf{p})} - \frac{s_q q_{\parallel}}{q_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \delta(sk_{\parallel} - s_p p_{\parallel} - s_q q_{\parallel}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

This equation tells us that we only have a nonlinear contribution when the wave polarities s_p and s_q are different. We recover here a well-known property of incompressible MHD: the nonlinear interactions are only due to counter-propagating Alfvén waves. This remark leads eventually to the following simplified form:

$$\partial_t q_{\Lambda}^s(\mathbf{k}) = \frac{\pi \epsilon^2}{2 b_0} \int \sum_{\Lambda_p, \Lambda_q} \left(\frac{\sin \psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (\Lambda k - \Lambda_p p + \Lambda_q q)^2 \quad (5.16)$$

$$q_{\Lambda_p}^{-s}(\mathbf{p}) \left[q_{\Lambda_q}^s(\mathbf{q}) - q_{\Lambda}^s(\mathbf{k}) \right] \delta(p_{\parallel}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

This result is exactly the same as in Galtier (2006b) (see in particular Appendix D) where the MHD limit was discussed in the more general context of Hall MHD (the difference of a factor 8 disappears after renormalization of the density tensor $q_{\Lambda}^s(\mathbf{k})$). Note that the comparison with Galtier *et al.* (2000) is not direct since the complex helicity basis was not used. The presence of $\delta(p_{\parallel})$ arises because of the three-wave frequency resonance condition. This means that in any triadic resonant interaction, there is always one wave that corresponds to a purely two-dimensional motion ($p_{\parallel} = 0$) whereas the two

others have equal parallel components ($p_{\parallel} = k_{\parallel}$). In other words, that means there is no nonlinear transfer along \mathbf{b}_0 and a cascade happens only in the perpendicular direction.

5.5. Large-scale dynamics: inertial waves

We consider the large-scale limit of (3.28) for left-handed ($\Lambda = -s$) fluctuations. Then, we have at the leading order:

$$\xi_{\Lambda}^s \rightarrow -\frac{skd}{2}, \quad (5.17)$$

$$\left(2 + \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_{\Lambda}^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2}\right)^2 \rightarrow \left(\frac{64}{k^2 p^2 q^2 d^6}\right)^2, \quad (5.18)$$

$$\omega_{\Lambda}^s \rightarrow \frac{2\Omega_0 s k_{\parallel}}{k} = \omega_I. \quad (5.19)$$

After introducing the previous expressions into (3.28), we obtain:

$$\partial_t q_{\Lambda}^s(\mathbf{k}) = \frac{\pi \epsilon^2}{4b_0^2} \int \sum_{\Lambda_p, \Lambda_q} \left(\frac{\sin \psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \frac{(\Lambda_q q - \Lambda_p p)^2}{p^2 q^2} \frac{k^2 \omega_{\Lambda}^{-\Lambda}}{k_{\parallel}^2} \quad (5.20)$$

$$q_{\Lambda}^{-\Lambda}(\mathbf{k}) q_{\Lambda_p}^{-\Lambda_p}(\mathbf{p}) q_{\Lambda_q}^{-\Lambda_q}(\mathbf{q}) \left[\frac{k^2 \omega_{\Lambda}^{-\Lambda}}{q_{\Lambda}^{-\Lambda}(\mathbf{k})} - \frac{p^2 \omega_{\Lambda_p}^{-\Lambda_p}}{q_{\Lambda_p}^{-\Lambda_p}(\mathbf{p})} - \frac{q^2 \omega_{\Lambda_q}^{-\Lambda_q}}{q_{\Lambda_q}^{-\Lambda_q}(\mathbf{q})} \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

This result is exactly the same as in Galtier (2003) provided that the density tensor is correctly renormalized.

5.6. Large-scale dynamics: magnetostrophic waves

The last limit that we shall consider is the large-scale one for right-handed ($\Lambda = s$) fluctuations. We have at leading order:

$$\xi_{\Lambda}^s \rightarrow -\frac{2s}{kd}, \quad (5.21)$$

$$\left(2 + \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} - \xi_{\Lambda}^{-s^2} - \xi_{\Lambda_p}^{-s_p^2} - \xi_{\Lambda_q}^{-s_q^2}\right)^2 \rightarrow 4, \quad (5.22)$$

$$\omega_{\Lambda}^s \rightarrow \frac{sk_{\parallel} k db_0}{2} = \omega_M. \quad (5.23)$$

After introducing the previous expressions into (3.28), we obtain:

$$\partial_t q_{\Lambda}^s(\mathbf{k}) = \frac{\pi \epsilon^2}{b_0^2} \int \sum_{\Lambda_p, \Lambda_q} \left(\frac{\sin \psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (\Lambda_p p - \Lambda_q q)^2 \frac{\omega_{\Lambda}^{\Lambda}}{k_{\parallel}^2} \quad (5.24)$$

$$q_{\Lambda}^{\Lambda}(\mathbf{k}) q_{\Lambda_p}^{\Lambda_p}(\mathbf{p}) q_{\Lambda_q}^{\Lambda_q}(\mathbf{q}) \left[\frac{\omega_{\Lambda}^{\Lambda}}{q_{\Lambda}^{\Lambda}(\mathbf{k})} - \frac{\omega_{\Lambda_p}^{\Lambda_p}}{q_{\Lambda_p}^{\Lambda_p}(\mathbf{p})} - \frac{\omega_{\Lambda_q}^{\Lambda_q}}{q_{\Lambda_q}^{\Lambda_q}(\mathbf{q})} \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

This system has never been analyzed before, however, it is similar to the electron MHD case (Galtier & Bhattacharjee 2003).

6. Exact solutions for the turbulent spectra

We shall derive the exact solutions of the weak turbulence equations in three different limits: the large and small wave number limits with in the latter case a distinction between

right and left polarizations. For that, we need to write the expression of the spectral density $q_\Lambda^s(\mathbf{k})$ in terms of explicit quantities like the kinetic and magnetic energies, the cross- and magnetic helicities. We inverse the system $(E^u, E^b, H^c, H^m)(q_\Lambda^s)$ and obtain:

$$q_\Lambda^s(\mathbf{k}) = \quad (6.1)$$

$$\frac{1}{2(\xi_\Lambda^{s^2} - \xi_\Lambda^{-s^2})} \left[\xi_\Lambda^{s^2} E^b(\mathbf{k}) - E^u(\mathbf{k}) + (\xi_\Lambda^s + \xi_\Lambda^{-s}) H^c(\mathbf{k}) + \Lambda(\xi_\Lambda^{s^2} - 1) k H^m(\mathbf{k}) \right].$$

The introduction of expression (6.1) into (3.28) leads to weak turbulence equations for E^u , E^b , H^c and H^m . However, since we are only interested by three asymptotic limits (Alfvén, inertial and magnetostrophic wave turbulence) for which we are able to derive the solutions, we may simplify the problem by taking the asymptotic values of the coefficients ξ_Λ^s (see Section 5).

6.1. Solutions for Alfvén wave turbulence

The small-scale limit of Alfvén wave turbulence is very well-known and has been analyzed in detail by Galtier *et al.* (2000). For an application to the dynamo it is not the most relevant limit since the magnetic energy is expected to be accumulated at the largest scales of the system. Therefore, we will not give details about this regime but only recall the main properties. In the small-scale limit ($kd \rightarrow +\infty$), for which terms like $\xi_\Lambda^{-s^2}$ tend to 1, an equipartition between the kinetic and magnetic energies is obtained and their dynamical equations tend to be identical. If we neglect the helicity contributions, the equation for the total energy gets reduce (see the derivation given in Galtier (2006b) where the helicity decomposition is used) and it is then possible to demonstrate that the axisymmetric bi-dimensional total energy spectrum follows the universal solution:

$$E(k_\perp, k_\parallel) \sim k_\perp^{-2} f(k_\parallel), \quad (6.2)$$

where f is an arbitrary function which traduces the dynamical decoupling of parallel planes in Fourier space. In other words, in Alfvén wave turbulence the cascade towards small-scales only happens in the perpendicular direction. This regime with its predictions has been observed in direct numerical simulations (Perez & Boldyrev 2008; Bigot *et al.* 2008; Bigot & Galtier 2011).

6.2. Solutions for inertial wave turbulence

When the small-scale limit is taken with only the left polarization retained, one arrives to the inertial wave turbulence regime which was derived analytically by Galtier (2003) and studied numerically by Bellet *et al.* (2006). Since $\xi_\Lambda^s \rightarrow 0$, we see immediately from relation (6.1) that the magnetic energy becomes negligible compared to the kinetic energy. Additionally, a simple analysis of equation (5.20) allows us to conclude that this turbulence becomes anisotropic. Indeed, if we assume that the nonlinear transfer is mainly the result of local interactions (*i.e.* equilateral triads $k \approx p \approx q$), then the resonance condition (3.26) simplifies to:

$$\frac{s_p - s}{ss_p q_\parallel} \approx \frac{s_p - s_q}{s_p s_q k_\parallel} \approx \frac{s - s_q}{ss_q p_\parallel}. \quad (6.3)$$

From equations (5.20), we see that only the interactions between two waves (\mathbf{p} and \mathbf{q}) with opposite polarities ($s = s_p = -s_q$ or $s = -s_p = s_q$; with $s = -\Lambda$) will contribute significantly to the nonlinear dynamics. It implies that either $q_\parallel \approx 0$ or $p_\parallel \approx 0$ which means that only a small transfer is allowed along Ω_0 . In other words, the local nonlinear interactions lead to anisotropic turbulence where small-scales are preferentially generated

perpendicularly to the external rotation axis. Note that this approximation is particularly well verified initially if the turbulence is mainly excited in a limited band of scales: then, by nature the nonlinear interactions will be local and will produce anisotropy. This short analysis allows us to consider the anisotropic limit of equation (5.20) for which $k_\perp \gg k_\parallel$. We obtain the following equations:

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \frac{E_k}{H_k} \right\} = & \quad (6.4) \\ \frac{\Omega^2 \epsilon^2}{4} \sum_{s_p s_q} \int \frac{s k_\parallel s_p p_\parallel}{k_\perp^2 p_\perp^2 q_\perp^2} \left(\frac{s_q q_\perp - s_p p_\perp}{\omega_k} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin \theta_q \\ & \left\{ \begin{aligned} & E_q (p_\perp E_k - k_\perp E_p) + (p_\perp s H_k / k_\perp - k_\perp s_p H_p / p_\perp) s_q H_q / q_\perp \\ & s k_\perp [E_q (p_\perp s H_k / k_\perp - k_\perp s_p H_p / p_\perp) + (p_\perp E_k - k_\perp E_p) s_q H_q / q_\perp] \end{aligned} \right\} \\ & \delta(s \omega_k + s_p \omega_p + s_q \omega_q) \delta(k_\parallel + p_\parallel + q_\parallel) dp_\perp dq_\perp dp_\parallel dq_\parallel, \end{aligned}$$

where $E_k \equiv E^u(k_\perp, k_\parallel)$ and $H_k \equiv H^k(k_\perp, k_\parallel)$ are respectively the axisymmetric bi-dimensional kinetic energy and kinetic helicity spectra, θ_q is the angle between the perpendicular wave vectors \mathbf{k}_\perp and \mathbf{p}_\perp in the triangle made with $(\mathbf{k}_\perp, \mathbf{p}_\perp, \mathbf{q}_\perp)$ and $\omega_k \simeq 2\Omega k_\parallel / k_\perp$. In equation (6.4) the integration over perpendicular wave numbers is such that the triangular relation, $\mathbf{k}_\perp + \mathbf{p}_\perp + \mathbf{q}_\perp = \mathbf{0}$, must be satisfied. The exact solutions of equations (6.4) were derived initially for a positive and constant kinetic energy flux (Galtier 2003); they read:

$$E_k \sim k_\perp^{-5/2} |k_\parallel|^{-1/2}, \quad (6.5)$$

$$H_k \sim k_\perp^{-3/2} |k_\parallel|^{-1/2}. \quad (6.6)$$

In a situation where the turbulence is dominated by a (forward) helicity flux, it is necessary to consider the equation for the kinetic helicity to derive the other exact power law solutions. If we seek stationary solutions in the power law form $E_k \sim k_\perp^n |k_\parallel|^m$ and $H_k \sim k_\perp^{\tilde{n}} |k_\parallel|^{\tilde{m}}$, then the constant helicity flux solutions are more general and read (Galtier 2014):

$$n + \tilde{n} = -4, \quad (6.7)$$

$$m + \tilde{m} = -1. \quad (6.8)$$

These solutions correspond to a positive helicity flux and thus a direct cascade. The cascade along the rotation axis being strongly reduced, the most important scaling law is therefore the one for the perpendicular wave numbers. It is remarkable to see that the exact solution (6.7) corresponds to the empirical law observed in a myriad of direct numerical simulations where the helicity transfer dominates the energy transfer (see e.g. Mininni & Pouquet 2009; Mininni *et al.* 2012). The domain of convergence of this family of solutions writes:

$$-3 < n + m < -2, \quad (6.9)$$

$$-2 < \tilde{n} + \tilde{m} < -1. \quad (6.10)$$

The spectral solutions of the inertial wave turbulence regime are at the border line of the domain of convergence. However, since the problem is strongly anisotropic and the inertial range in the parallel direction is strongly reduced with a cascade almost only in the perpendicular direction, we may neglect the inertial range in the parallel direction which

is equivalent to say $m = \tilde{m} = 0$. Then, we obtain a classical result of weak turbulence in the sense that the power law indices of the exact solutions (6.5)–(6.6) fall exactly at the middle of the domains of locality (6.9)–(6.10). In conclusion, we see that the turbulent spectra does not correspond necessarily to the so-called maximal helicity state which is a particular solution of the Schwarz inequality $H(\mathbf{k}) \leq kE(\mathbf{k})$ (here we consider directly the weak turbulence limit for which the polarization term (Cambon & Jacquin 1989) does not contribute) and for which $n = \tilde{n} - 1 = -5/2$. As the helicity transfer increases the power law indices n and \tilde{n} get closer. The condition of locality gives, however, a limit to this convergence namely $n = \tilde{n} = -2$.

6.3. Solutions for magnetostrophic wave turbulence

The small-scale limit of expression (3.28) can lead to the magnetostrophic wave turbulence equations if only the right polarization is retained. As for inertial wave turbulence, we may show from equation (5.20) that this turbulence becomes naturally anisotropic. Indeed, if we consider that the nonlinear transfer is mainly due to local interactions ($k \approx p \approx q$), the resonance condition (3.26) simplifies to:

$$\frac{s_p - s}{q_{\parallel}} \approx \frac{s_p - s_q}{k_{\parallel}} \approx \frac{s - s_q}{p_{\parallel}}. \quad (6.11)$$

From equations (5.24), we see that only the interactions between two waves (\mathbf{p} and \mathbf{q}) with opposite polarities ($s = s_p = -s_q$ or $s = -s_p = s_q$; with $s = \Lambda$) will contribute significantly to the nonlinear dynamics. It implies that either $q_{\parallel} \approx 0$ or $p_{\parallel} \approx 0$ which means that only a small transfer is allowed along Ω_0 . As for inertial wave turbulence, (i) the local nonlinear interactions lead to anisotropic turbulence where the cascade is preferentially generated perpendicularly to the external rotation axis, and (ii) the approximation is particularly well verified initially if the turbulence is mainly excited in a limited band of scales since then, by nature the nonlinear interactions will be local. From this discussion, it seems relevant to take the anisotropic limit ($k_{\perp} \gg k_{\parallel}$) of equation (5.24) which gives:

$$\begin{aligned} \partial_t \left\{ \begin{array}{c} E_k \\ H_k \end{array} \right\} = & \quad (6.12) \\ \frac{\epsilon^2}{16} \sum_{ss_p s_q} \int \frac{s_p p_{\perp} k_{\parallel} p_{\parallel}}{q_{\perp}} \left(\frac{s_q q_{\perp} - s_p p_{\perp}}{k_{\parallel}} \right)^2 (sk_{\perp} + s_p p_{\perp} + s_q q_{\perp})^2 \sin \theta_q & \\ \left\{ \begin{array}{c} sk_{\perp} [E_q(p_{\perp} E_k - k_{\perp} E_p)/(k_{\perp} p_{\perp} q_{\perp}) + s_q H_q (sH_k - s_p H_p)] \\ E_q(sH_k - s_p H_p)/q_{\perp} + s_q H_q(p_{\perp} E_k - k_{\perp} E_p)/(k_{\perp} p_{\perp}) \end{array} \right\} & \\ \delta(k_{\parallel} + p_{\parallel} + q_{\parallel}) \delta(sk_{\perp} k_{\parallel} + s_p p_{\perp} p_{\parallel} + s_q q_{\perp} q_{\parallel}) dp_{\perp} dq_{\perp} dp_{\parallel} dq_{\parallel}, & \end{aligned}$$

where $E_k \equiv E^b(k_{\perp}, k_{\parallel})$ and $H_k \equiv H^m(k_{\perp}, k_{\parallel})$ are respectively the axisymmetric bi-dimensional magnetic energy and magnetic helicity spectra and, as before, θ_q is the angle between the perpendicular wave vectors \mathbf{k}_{\perp} and \mathbf{p}_{\perp} in the triangle made with $(\mathbf{k}_{\perp}, \mathbf{p}_{\perp}, \mathbf{q}_{\perp})$. In equation (6.12) the integration over perpendicular wave numbers is such that the triangular relation, $\mathbf{k}_{\perp} + \mathbf{p}_{\perp} + \mathbf{q}_{\perp} = \mathbf{0}$, must be satisfied. To derive the exact solutions, we have to introduce the following power law forms for the spectra $E_k \sim k_{\perp}^n |k_{\parallel}|^m$ and $H_k \sim k_{\perp}^{\tilde{n}} |k_{\parallel}|^{\tilde{m}}$, and apply a bi-homogeneous conformal transform (Zakharov *et al.* 1992; Nazarenko 2011) which consists in doing the following manipulation on the wave numbers

p_\perp , q_\perp , p_\parallel and q_\parallel :

$$\begin{aligned} p_\perp &\rightarrow k_\perp^2/p_\perp, \\ q_\perp &\rightarrow k_\perp q_\perp/p_\perp, \\ |p_\parallel| &\rightarrow k_\parallel^2/|p_\parallel|, \\ |q_\parallel| &\rightarrow |k_\parallel||q_\parallel|/|p_\parallel|. \end{aligned} \tag{6.13}$$

This exercise for the energy equation gives the positive and constant energy flux solutions:

$$E_k \sim k_\perp^{-5/2} |k_\parallel|^{-1/2}, \tag{6.14}$$

$$H_k \sim k_\perp^{-7/2} |k_\parallel|^{-1/2}. \tag{6.15}$$

The same transform applied to the helicity equation extends the previous solutions to a family of solutions:

$$n + \tilde{n} = -6, \tag{6.16}$$

$$m + \tilde{m} = -1. \tag{6.17}$$

This family of solutions corresponds to a negative and constant magnetic helicity flux, hence the possible existence of an inverse cascade of helicity and the accumulation of magnetic energy at large-scales. Since the cascade along the uniform magnetic field is strongly reduced, the most important scaling law is therefore the one for the perpendicular wave numbers. The domain of convergence of these solutions writes:

$$-3 < n + m < -2, \tag{6.18}$$

$$-4 < \tilde{n} + \tilde{m} < -3. \tag{6.19}$$

We see that with the previous solutions (obtained from the energy or the helicity equations) we are at the border line of the domain of convergence. However, we also know that this problem is strongly anisotropic and the inertial range in the parallel direction is strongly reduced with a cascade almost only in the perpendicular direction. Actually, if we neglect the inertial range in the parallel direction (which is equivalent to say $m = \tilde{m} = 0$) we obtain again – like for the inertial wave turbulence regime – a classical result of weak turbulence in the sense that the power law indices of the exact solutions (6.14)–(6.15) fall exactly at the middle of the domains of locality (6.18)–(6.19). Note that the solutions found do not allow a crossing of the spectra since the case $n = \tilde{n} = -3$ appears as an asymptotic limit. Note also that the classical phenomenology presented in Section 4 gives the particular asymptotic solution $\tilde{n} = -3$. It is only through a deep mathematical treatment that this family of solutions may be discovered. This situation is also found for the inertial wave turbulence regime for which many papers have been devoted but where no consistent anisotropic phenomenology has been proposed. For that reason, these exact solutions may be qualified as highly non-trivial. Finally, it is interesting to remark that the process of inverse cascade described here is limited in scales since the basic assumption made for the analysis is that $k_\perp \gg k_\parallel$. When this condition is broken (with e.g. $k_\perp \ll k_\parallel$), the previous local analysis made on the resonance condition becomes irrelevant and the theoretical predictions not possible.

7. Discussion

In this paper I have developed a weak turbulence theory for rotating MHD under the presence of a parallel uniform magnetic field. The theory is expected to be relevant for the magnetostrophic dynamo with applications to Earth and giant planets for which

a small ($\sim 10^{-6}$) Rossby number is expected. An important question which may be investigated is the mechanism of regeneration of a large-scale magnetic field through an inverse cascade of hybrid helicity. A key length scale in this problem is the magneto-inertial length d which indicates the basin of attraction for the dynamics. Basically, if the scales considered are larger than d (in other words if $kd < 1$), we fall in the inertial or magnetostrophic wave turbulence regime, the precise localization being determined by the nature of the polarization (left or right respectively). If, however, the scales are smaller than d ($kd > 1$) and if the condition for weak turbulence are still satisfied (with a wave period much smaller than the eddy-turn-over time, otherwise the turbulence is strong), then we fall in the Alfvén wave turbulence regime. It is interesting to note that the magnetostrophic regime – also called strong-field regime – is driven by a nonlinear equation (2.27) similar to a well-known system in plasma physics called electron MHD (Kingsep *et al.* 1990) which finds applications, e.g. in space plasmas (Galtier 2006*a*).

By using a complex helicity decomposition, the asymptotic weak turbulence equations have been derived which describe the long-time behavior of weakly dispersive interacting waves *via* three-wave processes. For magnetostrophic wave turbulence, the theory predicts that the magnetic energy is asymptotically larger than the kinetic energy when one goes to large-scales, whereas it is the inverse for inertial wave turbulence. The analysis of the resonance conditions has been used to prove the anisotropic nature of the nonlinear transfer with a stronger cascade perpendicular than parallel to the rotating axis. Then, the reduced forms of the general equations of weak turbulence have been obtained in the three relevant limits discussed above with their exact power law solutions after the application of the Kuznetsov–Zakharov transform (see figure 4). The large-scale (magnetostrophic and inertial) solutions can be highly non-trivial in the sense that the classical anisotropic phenomenology is only able to catch the correct scaling for the constant energy flux solutions which are dimensionally compatible with a maximal helicity state. The solutions for the constant (magnetic or kinetic) helicity flux are, however, not recovered with a consistent phenomenology. The non triviality resides in an entanglement relation which implies the energy and helicity spectra power law indices. At large-scales ($kd < 1$), whereas a direct cascade of kinetic helicity is expected which is well observed in direct numerical simulations of pure rotating hydrodynamic turbulence (see e.g. Mininni & Pouquet 2009; Mininni *et al.* 2012), an inverse cascade of magnetic helicity is predicted. Since the magnetostrophic wave turbulence regime is similar to the electron MHD one where an inverse cascade has already been observed in direct numerical simulations (Shaikh & Zank 2005; Cho 2011) we may think that it is a reasonable prediction. Then, in the context of the dynamo problem the main question is: at which scale k_f the system is driven? Indeed, if the forcing scales is such that $k_f d < 1$ we fall in the large-scale regime (magnetostrophic basin of attraction; see figure 4) and the dynamo mechanism may happen through an inverse cascade of hybrid helicity which is dominated by the magnetic helicity. However, if that scale is such that $k_f d > 1$, then we fall in the small-scale regime (Alfvén basin of attraction) and the regeneration of the magnetic field becomes more difficult since the hybrid helicity is dominated by the cross-helicity which cascades in the forward (to small-scales) direction (Galtier *et al.* 2000). It is important to recall that the magnetic helicity is *not* an inviscid invariant in the weak (non rotating) MHD turbulence regime where a uniform magnetic field is present; the question of the regeneration of a large-scale magnetic field needs therefore a new ingredient like the Coriolis force to be relevant.

The present theory may be useful to better understand the magnetostrophic dynamo with applications to Earth and giant planets. Although our theory is a crude model for such a problem (for example, we assume a magnetic Reynolds numbers large enough

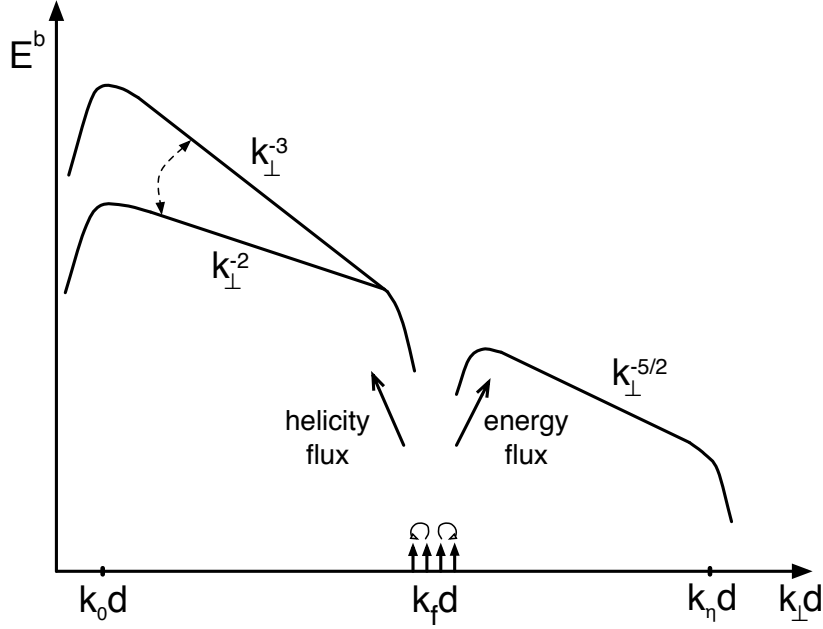


FIGURE 4. Magnetic energy spectrum in the weak magnetostrophic turbulence regime ($kd < 1$) and with a (e.g. convective) forcing applied in a range of intermediate scales k_f . While the direct energy cascade gives a unique scaling, the inverse helicity cascade may lead to a family of solutions confined between k_{\perp}^{-2} and k_{\perp}^{-3} due to the entanglement of the helicity and energy. In practice, the inertial ranges are limited by the largest scale of the system k_0 (e.g. the size of the outer core) and the dissipative scale (e.g. the magnetic one) k_{η} .

for the development of an extended inertial range and we do not include the geometry effects with boundary conditions), it is believed that the dynamics obtained here at asymptotically small Rossby number opens new perspectives. For example, in the case of the outer core of Earth a rough evaluation of the magneto-inertial length gives $d \approx 1\text{km}$ (Finlay *et al.* 2010). If we consider that the forcing due to convection has a typical length scale of $1/k_f \approx 100\text{km}$ then the conditions for an inverse cascade are satisfied. Another question is about the surprising axisymmetry of planets like Earth, Jupiter or Saturn where the rotation and magnetic axes are close and even almost perfect for Saturn. The present turbulence theory gives a possible answer. Indeed, the rotating MHD equations in presence of a uniform magnetic field have in general only one inviscid invariant, the total energy. It is only when the rotation and magnetic axes are aligned that a second inviscid invariant appears, namely the hybrid helicity. It is precisely this second invariant which can generate a turbulent dynamo through an inverse cascade. We may believe that as long as the angle θ between $\mathbf{\Omega}_0$ and \mathbf{b}_0 remains reasonably small the inverse cascade may still operate. According to this remark, it is not surprising that a strong alignment, with $\theta \leq 10^\circ$, is generally observed for the previous magnetized planets. The initial phase of the dynamo has not been discussed until now but it deserves a short discussion. Since in absence of a uniform magnetic field the magnetic helicity is an inviscid invariant of rotating MHD, an inverse cascade may happen. This mechanism is, however, under the influence of the Coriolis force which renders the dynamics anisotropic. Then, we may expect the generation of a large-scale magnetic preferentially aligned with the rotation axis. After this initial phase, it seems then natural to consider the regime described in the present paper.

Appendix A. Useful relationships

From the quantity:

$$\xi_\Lambda^s = \frac{-skd}{(-s\Lambda + \sqrt{1 + k^2 d^2})}, \quad (\text{A } 1)$$

it is possible to derive the following useful identities:

$$\xi_\Lambda^s \xi_\Lambda^{-s} = -1, \quad (\text{A } 2)$$

$$\xi_{-\Lambda}^{-s} = -\xi_\Lambda^s, \quad (\text{A } 3)$$

$$\xi_\Lambda^s + \xi_\Lambda^{-s} = -\frac{2}{\Lambda kd}, \quad (\text{A } 4)$$

$$\xi_\Lambda^s - \xi_\Lambda^{-s} = -\frac{2s}{kd} \sqrt{1 + k^2 d^2}, \quad (\text{A } 5)$$

$$1 - \xi_\Lambda^{s^2} = \frac{2\Lambda\Omega_0}{b_0 k} \xi_\Lambda^s. \quad (\text{A } 6)$$

We also have the remarkable relations:

$$\omega_s^s \omega_{-s}^s = (k_\parallel \mathbf{b}_0)^2, \quad (\text{A } 7)$$

$$\omega_s^{s^2} \leq (k_\parallel \mathbf{b}_0)^2 \leq \omega_{-s}^{s^2}. \quad (\text{A } 8)$$

Appendix B. Helicity decomposition

The projection of the Fourier transform of the original vectors $\mathbf{u}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ on the helicity basis gives:

$$\mathbf{u}_\mathbf{k} = \sum_\Lambda \mathcal{U}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda, \quad (\text{B } 1)$$

$$\mathbf{b}_\mathbf{k} = \sum_\Lambda \mathcal{B}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda. \quad (\text{B } 2)$$

If we inverse the system, we find the following relations for the velocity components:

$$\mathcal{U}_+(\mathbf{k}) = \frac{1}{2kk_\perp} [k_x k_\parallel u_x + k_y k_\parallel u_y - k_\perp^2 u_z + ik(k_y u_x - k_x u_y)], \quad (\text{B } 3)$$

$$\mathcal{U}_-(\mathbf{k}) = \frac{1}{2kk_\perp} [k_x k_\parallel u_x + k_y k_\parallel u_y - k_\perp^2 u_z - ik(k_y u_x - k_x u_y)]. \quad (\text{B } 4)$$

Similar relations are found for the magnetic field. Note that such helicity decomposition cannot be applied for the modes $k_\perp = 0$.

Appendix C. Derivation of the weak turbulence equations

The starting point of the derivation of the weak turbulence equations is the fundamental equation (3.19). We write successively equations for the second and third-order moments:

$$\begin{aligned} \partial_t \langle a_\Lambda^s a_{\Lambda'}^{s'} \rangle = \\ \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} M_{\substack{\Lambda \Lambda_p \Lambda_q \\ s s_p s_q}}^{s s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda'}^{s'} \rangle e^{-i\Omega_{pq,k} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q} \end{aligned} \quad (\text{C } 1)$$

$$+ \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} M_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q s' s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s \rangle e^{-i\Omega_{pq, k'} t} \delta_{pq, k'} d\mathbf{p} d\mathbf{q},$$

and:

$$\begin{aligned} \partial_t \langle a_{\Lambda}^s a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle = & \quad (C 2) \\ & \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} M_{-k p q}^{\Lambda \Lambda_p \Lambda_q s s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle e^{-i\Omega_{pq, k} t} \delta_{pq, k} d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} M_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q s' s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s a_{\Lambda''}^{s''} \rangle e^{-i\Omega_{pq, k'} t} \delta_{pq, k'} d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon d^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} M_{-k'' p q}^{\Lambda'' \Lambda_p \Lambda_q s'' s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s a_{\Lambda'}^{s'} \rangle e^{-i\Omega_{pq, k''} t} \delta_{pq, k''} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

We shall write an asymptotic closure (Nazarenko 2011) for our system. For that, we basically need to write the fourth-order moment in terms of a sum of the fourth-order cumulant plus products of second order ones. The asymptotic closure depends on two ingredients: the first is the degree to which the linear waves interact to randomize phases; the second relies on the fact that the nonlinear regeneration of the third-order moment by the fourth-order moment in equation (C 2) depends more on the product of the second order moments than it does on the fourth order cumulant. The fourth-order moment decomposes into the sum of three products of second-order moments, and a fourth-order cumulant. The latter does not contribute to secular behavior, and among the other products one is absent because of the homogeneity assumption. If we use the symmetric relations (3.21)–(3.24) and perform wavevector integrations, summations over polarities and time integration, then equation (C 2) becomes:

$$\begin{aligned} \langle a_{\Lambda}^s a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle = & \frac{\epsilon d^2}{16} \Delta(\Omega_{kk'k''}) \delta_{kk'k''} \\ & \{ \left[\frac{\xi_{\Lambda''}^{-s''} - \xi_{\Lambda'}^{-s'}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \left(M_{k k' k''}^{\Lambda \Lambda' \Lambda'' s s' s''} \right)^* + \frac{\xi_{\Lambda'}^{-s'} - \xi_{\Lambda''}^{-s''}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \left(M_{k k'' k'}^{\Lambda \Lambda'' \Lambda' s s'' s'} \right)^* \right] q_{\Lambda'}^{s'} q_{\Lambda''}^{s''} \\ & + \\ & \left[\frac{\xi_{\Lambda''}^{-s''} - \xi_{\Lambda}^{-s}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left(M_{k' k k''}^{\Lambda' \Lambda \Lambda'' s' s s''} \right)^* + \frac{\xi_{\Lambda}^{-s} - \xi_{\Lambda''}^{-s''}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left(M_{k' k'' k}^{\Lambda' \Lambda'' \Lambda s' s'' s} \right)^* \right] q_{\Lambda}^s q_{\Lambda''}^{s''} \\ & + \\ & \left[\frac{\xi_{\Lambda}^{-s} - \xi_{\Lambda'}^{-s'}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} \left(M_{k'' k' k}^{\Lambda'' \Lambda' \Lambda s'' s' s} \right)^* + \frac{\xi_{\Lambda'}^{-s'} - \xi_{\Lambda}^{-s}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} \left(M_{k'' k k'}^{\Lambda'' \Lambda \Lambda' s'' s s'} \right)^* \right] q_{\Lambda'}^{s'} q_{\Lambda}^s \} , \end{aligned} \quad (C 3)$$

where:

$$\Delta(\Omega_{kk'k''}) = \int_0^t e^{i\Omega_{kk'k''}t'} dt' = \frac{e^{i\Omega_{kk'k''}t} - 1}{i\Omega_{kk'k''}}. \quad (\text{C } 4)$$

The introduction of symmetric relations (3.21)–(3.24) into (C 3) allows us to simplify further the previous equation; one obtains:

$$\langle a_\Lambda^s a_\Lambda^{s'} a_\Lambda^{s''} \rangle = \frac{\epsilon d^2}{8} \Delta(\Omega_{kk'k''}) \delta_{kk'k''} \left(M_{k k' k''}^{\Lambda \Lambda' \Lambda'' s s' s''} \right)^* \quad (\text{C } 5)$$

$$\left[\frac{\xi_{\Lambda'}^{-s''} - \xi_{\Lambda'}^{-s'}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} q_{\Lambda'}^{s'} q_{\Lambda''}^{s''} + \frac{\xi_\Lambda^{-s} - \xi_{\Lambda''}^{-s''}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} q_\Lambda^s q_{\Lambda''}^{s''} + \frac{\xi_{\Lambda'}^{-s'} - \xi_\Lambda^{-s}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} q_\Lambda^s q_{\Lambda'}^{s'} \right].$$

We insert expression (C 5) into equation (C 1); it leads to:

$$\begin{aligned} \partial_t q_\Lambda^s(\mathbf{k}) = & \quad (\text{C } 6) \\ & \frac{\epsilon^2 d^4}{128} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} \left| M_{-k p q}^{\Lambda \Lambda_p \Lambda_q s s_p s_q} \right|^2 \Delta(\Omega_{pq,k}) e^{-i\Omega_{pq,k}t} \delta_{pq,k} \\ & \left[\frac{\xi_\Lambda^{-s} - \xi_{\Lambda_q}^{-s_q}}{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda_p}^{-s_p}} q_\Lambda^s q_{\Lambda_q}^{s_q} + \frac{\xi_{\Lambda_p}^{-s_p} - \xi_\Lambda^{-s}}{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_q}^{-s_q}} q_\Lambda^s q_{\Lambda_p}^{s_p} + \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon^2 d^4}{128} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left| M_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q s' s_p s_q} \right|^2 \Delta(\Omega_{pq,k'}) e^{-i\Omega_{pq,k'}t} \delta_{pq,k'} \\ & \left[\frac{\xi_{\Lambda'}^{-s'} - \xi_{\Lambda_q}^{-s_q}}{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda_p}^{-s_p}} q_{\Lambda'}^{s'} q_{\Lambda_q}^{s_q} + \frac{\xi_{\Lambda_p}^{-s_p} - \xi_{\Lambda'}^{-s'}}{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_q}^{-s_q}} q_{\Lambda'}^{s'} q_{\Lambda_p}^{s_p} + \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q}. \end{aligned}$$

The long-time behavior of the weak turbulence equation (C 6) is given by the Riemman-Lebesgue Lemma which tells us that, for $t \rightarrow +\infty$, we have:

$$e^{-ixt} \Delta(x) = \Delta(-x) \rightarrow \pi \delta(x) - i\mathcal{P}(1/x), \quad (\text{C } 7)$$

where \mathcal{P} is the principal value of the integral. The two terms of equation (C 6) are complex conjugated therefore if in the second term we replace the dummy integration variables \mathbf{p}, \mathbf{q} , by $-\mathbf{p}, -\mathbf{q}$, we can simplify further equation (C 6) since, in particular, principal value terms compensate exactly. Finally, we obtain the weak turbulence equation:

$$\begin{aligned} \partial_t q_\Lambda^s(\mathbf{k}) = & \quad (\text{C } 8) \\ & \frac{\pi \epsilon^2 d^4}{64} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} \left| M_{-k p q}^{\Lambda \Lambda_p \Lambda_q s s_p s_q} \right|^2 \delta(\Omega_{k,pq}) \delta_{k,pq} \\ & \left[\frac{\xi_\Lambda^{-s} - \xi_{\Lambda_q}^{-s_q}}{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda_p}^{-s_p}} q_\Lambda^s q_{\Lambda_q}^{s_q} + \frac{\xi_{\Lambda_p}^{-s_p} - \xi_\Lambda^{-s}}{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_q}^{-s_q}} q_\Lambda^s q_{\Lambda_p}^{s_p} + \frac{\xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q}, \end{aligned}$$

where:

$$\left| M \begin{array}{c} \Lambda \Lambda_p \Lambda_q \\ s_p s_q \\ -k p q \end{array} \right|^2 = \left(\frac{\sin \psi_k}{k} \right)^2 k^2 p^2 q^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2$$

$$\xi_\Lambda^2 \xi_{\Lambda_p}^2 \xi_{\Lambda_q}^2 \left(2 + \xi_\Lambda^{-2} \xi_{\Lambda_p}^{-2} \xi_{\Lambda_q}^{-2} - \xi_\Lambda^{-2} - \xi_{\Lambda_p}^{-2} - \xi_{\Lambda_q}^{-2} \right)^2.$$

The last step that we have to follow to obtain the same expression as (3.28) is to include the resonance relations (3.26) into the previous equations.

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